PMATH 347 Groups and Rings, Assignment 5 Due Tues July 16, 11:00 pm

1: (a) Let $H = \big\{(1), (12)(34), (13)(24), (14)(23)\big\} \leq S_4$. Show that $H \trianglelefteq S_4$ and determine which of the two groups $\mathbb{Z}_6$ and S_3 is isomorphic to S_4/H.

(b) Let $H = \big\langle (2,-1), (2,3) \big\rangle \leq \mathbb{Z}^2$. Show that $\big|\mathbb{Z}^2/H\big| = 8$, determine which of the three groups $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ or ${\mathbb{Z}_2}^3$ is isomorphic to $\mathbb{Z}^2/H$, and find a surjective group homomorphism ϕ from $\mathbb{Z}^2$ to one of these three groups with $\text{Ker}(\phi) = H$.

2: (The Second Isomorphism Theorem) Let G be a group and let $H, K \leq G$.

(a) Show that $HK \leq G \iff HK = KH$.

(b) Show that if $K \trianglelefteq G$ then $K \cap H \trianglelefteq H$, $KH \leq G$ and $K \trianglelefteq KH$.

(c) Show that if $K \trianglelefteq G$ then $H/(K \cap H) \cong KH/K$.

(d) Show that (even if $K \ntrianglelefteq G$) we have $|H|\,|K| = |KH|\,|K \cap H|$ (you may suppose that G is finite).

3: (a) (The Normalizer/Centralizer Theorem) Let G be a group and let $H \leq G$. Recall that the **centralizer** of H in G is the group $C(H) = C_G(H) = \big\{a \in G \big| ax = xa \text{ for all } x \in H\big\} \leq G$ and the **normalizer** of H in G is the group $N(H) = N_G(H) = \big\{a \in G \big| aH = Ha\big\} \leq G$. Show that $C(H) \trianglelefteq N(H)$ and that $N(H)/C(H)$ is isomorphic to a subgroup of $\text{Aut}(H)$.

(b) (The Orbit/Stabilizer Theorem) Let A be a nonempty set and let G be a finite subgroup of $\text{Perm}(A)$. For $a \in A$, the **orbit** of a is the set $\text{Orb}(a) = \big\{\sigma(a) \,\big|\, \sigma \in G\big\} \subseteq A$, and the **stabilizer** of a is the set $\text{Stab}(a) = \big\{\sigma \in G \,\big|\, \sigma(a) = a\big\}$. Show that for all $a \in A$, we have $\text{Stab}(a) \leq G$ and $|G| = \big|\text{Orb}(a)\big|\,\big|\text{Stab}(a)\big|$.

4: In this problem, when R is a ring and $X \subseteq R$, $\langle X \rangle$ denotes the ideal in R generated by X.

(a) Find the number of elements in $\mathbb{Z}^2/\langle(3,1)\rangle$.

(b) Find the number of elements in $\mathbb{Z}[i]/\langle 3+i \rangle$.

(c) Determine whether $\mathbb{Z}_5[i]/\langle 2+i \rangle \cong \mathbb{Z}_5$.