PMATH 336 Introduction to Group Theory, Solutions to the Term Test, Spring 2024

1: Use the following list of powers of 3 in U_{19} to help solve the problems below.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
3^k	1	3	9	8	5	15	7	2	6	18	16	10	11	14	4	12	17	13	1

(a) Find the number of elements of each order in U_{19}.

Solution: From the list of powers, we see that U_{19} is cyclic with $U_{19} = \langle 3 \rangle$ and $\varphi(19) = |U_{19}| = 18$. The divisors of 18 are 1, 2, 3, 6, 9 and 18, so the number of elements of each order is as follows:.

order	1	2	3	6	9	18
#	1	1	2	2	6	6

(b) List all of the elements of order 9 in U_{19}.

Solution: Note that $|9| = |3^2| = 9$ and $U_9 = \{1, 2, 4, 5, 7, 8\}$, and so the elements of order 9 are the generators of $\langle 9 \rangle = \langle 3^2 \rangle$ which are the elements $9^1 = 3^2 = 9$, $9^2 = 3^4 = 5$, $9^4 = 3^8 = 6$, $9^5 = 3^{10} = 16$, $9^7 = 3^{14} = 4$ and $9^8 = 3^{16} = 17$. Listed in increasing order, the elements of order 9 are 4, 5, 6, 9, 16, 17.

(c) Solve $x^8 = 6$ for $x \in U_{19}$.

Solution: Write $x = 3^k$. Then $x^8 = 6 \iff 3^{8k} = 3^8 \iff 8k = 8 \bmod 18 \iff 4k = 4 \bmod 9 \iff k = 1 \bmod 9 \iff k = 1 \text{ or } 10 \bmod 18 \iff x = 3^1 \text{ or } 3^{10} \iff x = 3 \text{ or } 16$.

2: (a) Write out the multiplication table for D_3.

Solution: Here is the table.

	I	R_1	R_2	F_0	F_1	F_2
I	I	R_1	R_2	F_0	F_1	F_2
R_1	R_1	R_2	I	F_1	F_2	F_0
R_2	R_2	I	R_1	F_2	F_0	F_1
F_0	F_0	F_2	F_1	I	R_2	R_1
F_1	F_1	F_0	F_2	R_1	I	R_2
F_2	F_2	F_1	F_0	R_2	R_1	I

(b) Solve $F_4X^3 = X\,F_2$ for $X \in D_6$.

Solution: When $X = R_k$ we have

$$\begin{aligned} F_4X^3 = X\,F_2 &\iff F_4R_{3k} = R_kF_2 \iff F_{4-3k} = F_{k+2} \\ &\iff 4-3k = k+2 \bmod 6 \iff 4k = 2 \bmod 6 \iff k \in \{2,5\} \end{aligned}$$

and when $X = F_k$ we have

$$\begin{aligned} F_4X^3 = X\,F_2 &\iff F_4F_k = F_kF_2 \iff R_{4-k} = R_{k-2} \\ &\iff 4-k = k-2 \bmod 6 \iff 2k = 0 \bmod 6 \iff k \in \{0,3\}. \end{aligned}$$

Thus the solutions are $X = R_2$, R_5, F_0 and F_3.

(c) Each element of D_4 is a matrix in $O(2,\mathbb{R})$. List the elements in D_4 as matrices.

Solution: We have

$$\begin{aligned} D_4 &= \{I, R_1, R_2, R_3, F_0, F_1, F_2, F_3\} \\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\} \end{aligned}$$

3: (a) Let $\alpha = (165347)(26534) \in S_7$. Write α as a product of disjoint cycles and find $|\alpha|$.

Solution: $\alpha = (1637)(254)$ and $|\alpha| = 12$.

(b) Let $\beta = (152374) \in S_7$. Write β^{-2} as a product of disjoint cycles.

Solution: $\beta^{-1} = (147325)$ and $\beta^{-2} = (172)(354)$.

(c) Let $\gamma = (1574)(2435) \in S_7$. Write γ as a product of 2-cycles and find $(-1)^\gamma$.

Solution: One solution is $\gamma = (14)(17)(15)(25)(23)(24)$, and we have$(-1)^\gamma = (-1)^6 = 1$.

(d) Determine the number of elements of order 6 in S_7.

Solution: The elements of order 6, when written as products of disjoint cycles, are of one of the forms $(abc)(de)$, $(abc)(de)(fg)$ or $(abcdef)$. The number of elements of the form $(abc)(de)$ is $\binom{7}{3} \cdot 2 \cdot \binom{4}{2} = 35 \cdot 2 \cdot 6 = 420$, the number of elements of the form $(abc)(de)(fg)$ is equal to $\binom{7}{3} \cdot 2 \cdot 3 = 35 \cdot 2 \cdot 3 = 210$, and the number of elements of the form $(abcdef)$ is equal to $\binom{7}{6} \cdot 5! = 7 \cdot 120 = 840$. Thus, altogether there are 420+210+840=1470 elements of order 6 in S_7.

4: (a) Find $\left|GL_2(\mathbb{Z}_3)\right|$.

Solution: Recall that when p is prime we have $\left|GL_n(\mathbb{Z}_p)\right| = (p^n-1)(p^n-p)(p^n-p^2)\cdots(p^n-p^{n-1})$, and so in particular $\left|GL_2(\mathbb{Z}_3)\right| = (3^2-1)(3^2-3) = 8\cdot 6 = 48$.

(b) Solve $X^2 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ for $X \in GL_2(\mathbb{Z}_3)$.

Solution: Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $X^2 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \iff \begin{pmatrix} a^2+bc & (a+d)b \\ (a+d)c & d^2+bc \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. Note that $(a+d)b = 2 \Longrightarrow b \neq 0$ and $(a+d)c = 1 \Longrightarrow c \neq 0$, so we have $bc \neq 0$, and then $a^2+bc = 0 \Longrightarrow a^2 \neq 0 \Longrightarrow a^2 = 1$ (since in $\mathbb{Z}_3$, $1^1 = 2^2 = 1$). Similarly, since $bc \neq 0$, $d^2+bc = 0 \Longrightarrow d^2 \neq 0 \Longrightarrow d^2 = 1$. Also note that since $a^2 = 1$, $a^2+bc = 0 \Longrightarrow bc = 2$. We now consider each of the two cases $b = 1$ and $b = 2$.

If $b = 1$ then $bc = 2 \Longrightarrow c = 2$ and then $(a+d)b = 2 \Longrightarrow a+d = 2 \Longrightarrow a = d = 1$ (since $a \neq 0$ and $d \neq 0$). If $b = 2$ then $bc = 2 \Longrightarrow c = 1$ and then $(a+d)b = 2 \Longrightarrow a+d = 1 \Longrightarrow a = d = 2$. Thus

$$X \in \left\{ \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \right\}.$$

(c) List all the elements in $SO_2(\mathbb{Z}_3)$ and determine whether $SO_2(\mathbb{Z}_3)$ is cyclic.

Solution: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $A \in O_2(\mathbb{Z}_3) \iff A^TA = I \iff \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In $\mathbb{Z}_3$ we have $0^2 = 0$, $1^2 = 2^2 = 1$ and so $a^2+c^2 = 1 \iff \big((a^2 = 0 \text{ and } c^2 = 1) \text{ or } (a^2 = 1 \text{ and } c^2 = 0)\big) \iff (a,c) = (0,1)$, $(0,2)$, $(1,0)$ or $(2,0)$. Similarly, $b^2+d^2 = 0 \iff (b,d) = (0,1)$, $(0,2)$, $(1,0)$ or $(2,0)$. Now suppose that $a^2+c^2 = 1$ and $b^2+d^2 = 1$. When $(a,c) = (0,1)$ or $(0,2)$, $ab+cd = 0 \iff (b,d) = (1,0)$ or $(2,0)$, and when $(a,c) = (1,0)$ or $(2,0)$, $ab+cd = 0 \iff (b,d) = (0,1)$ or $(0,2)$. Thus

$$O_2(\mathbb{Z}_3) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \right\}$$

and so

$$SO_2(\mathbb{Z}_3) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \right\}.$$

Note that $\left\langle \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \right\}$, and so $SO_2(\mathbb{Z}_3)$ is cyclic.