PMATH 336 Intro to Group Theory, Solutions to Assignment 7

In this assignment, T_u denotes the translation by the vector u, $R_{p,\theta}$ denotes the rotation about the point p by the angle θ, F_L denotes the reflection in the line L and $G_{u,L}$ denotes the glide reflection $G_{u,L} = T_u F_L = F_L T_u$ where u is parallel to L.

1: (a) Let $a = (2,1)$, $b = (3,3)$ and $c = (-1,5)$. Find the image of the triangle with vertices at a, b and c under the isometry $G_{(-4,4),x+y=3} F_{x-3y=4} R_{(5,2),\frac{\pi}{2}}$.

Solution: Using a picture, or using algebraic formulas for the isometries, we have

$$G_{(-4,4),x+y=3} F_{x-3y=4} R_{(5,2),\frac{\pi}{2}}(2,1) = G_{(-4,4),x+y=3} F_{x-3y=4}(6,-1) = G_{(-4,4),x+y=3}(5,2) = (-3,2)$$
$$G_{(-4,4),x+y=3} F_{x-3y=4} R_{(5,2),\frac{\pi}{2}}(3,3) = G_{(-4,4),x+y=3} F_{x-3y=4}(4,0) = G_{(-4,4),x+y=3}(4,0) = (-1,3)$$
$$G_{(-4,4),x+y=3} F_{x-3y=4} R_{(5,2),\frac{\pi}{2}}(-1,5) = G_{(-4,4),x+y=3} F_{x-3y=4}(2,-4) = G_{(-4,4),x+y=3}(0,2) = (-3,7)$$

So the image of triangle abc is the triangle with vertices at $(-3,2)$, $-1,3)$ and $(-3,7)$.

(b) Let L be the line $2x-3y=1$. Find the equation of the line M such that $F_M F_L = T_{(-2,3)}$.

Solution: The line M is the line obtained by translating the line L by $\frac{1}{2}(-2,3) = \left(-1,\frac{3}{2}\right)$. The line L passes through the point $(2,1)$ so the line M passes through the point $(2,1) + \left(-1+\frac{3}{2}\right) = \left(1,\frac{5}{2}\right)$. Thus M is the line through $\left(1,\frac{5}{2}\right)$ parallel to L, so M has equation $4x-6y+11=0$.

(c) Let L be the line $2x-3y=1$. Find the equation of the line N such that $F_N F_L = R_{(2,1),90^\circ}$.

Solution: The line N is the line obtain by revolving the line L by 45° about the point $(2,1)$. Notice that the points $(2,1)$, $(5,3)$, $(3,6)$ and $(0,4)$ form a square, so the line N is the diagonal which passes through $(2,1)$ and $(3,6)$. Thus N is the line $y=5x-9$.

2: (a) Express the isometry $R_{(4,4),90^\circ} F_{x+3y=6}$ as a glide reflection.

Solution: Let $S = R_{(4,4),90^\circ} F_{x+3y=6}$. Choose $a=(0,2)$ and $b=(3,1)$ (we could have chosen any two points a and b). We have

$$S(a) = R_{(4,4,),90^\circ} F_{x+3y=6}(0,2) = R_{(4,4),90^\circ}(0,2) = (6,0),$$
$$S(b) = R_{(4,4),90^\circ} F_{x+3y=6}(3,1) = R_{(4,4),90^\circ}(3,1) = (7,3).$$

The midpoint of a and $S(a)$ is $(3,1)$ and the midpoint of b and $S(b)$ is $(5,2)$. The reflection line L passes through these two midpoints, so L is the line $x-2y=1$. The translation vector is the vector $u = S(a) - F_L(a) = (6,0)-(2,-2) = (4,2)$. Thus $S = G_{(4,2),x-2y=1}$.

(b) Express the isometry $F_{y=3x} T_{(-2,3)} G_{(2,1),x+2=2y}$ as a single rotation.

Solution: Let $S = F_{y=3x} T_{(-2,3)} G_{(2,1),x+2=2y}$. Choose $a=(0,1)$ and $b=(2,2)$. We have

$$S(a) = F_{y=3x} T_{(-2,3)} G_{(2,1),x+2=2y}(0,1) = F_{y=3x} T_{(-2,3)}(2,2) = F_{y=3x}(0,5) = (3,4),$$
$$S(b) = F_{y=3x} T_{(-2,3)} G_{(2,1),x+2=2y}(2,2) = F_{y=3x} T_{(-2,3)}(4,3) = F_{y=3x}(2,6) = (2,6).$$

The rotation point p is equidistant from a and $S(a)$ so it lies on the perpendicular bisector of a and $S(a)$. Similarly, it lies on the perpendicular bisector of b and $S(b)$. The perpendicular bisector of a and $S(a)$ is the line $x+y=4$, and the perpendicular bisector of b and $S(b)$ is the line $y=4$. The rotation point is the point of intersection of these two lines, which is the point $p=(0,4)$. The rotation angle θ is the angle from the vector $b-a=(2,1)$ to the vector $S(b)-S(a) = (-1,2)$, that is $\theta = 90^\circ$. Thus $S = R_{(0,4),90^\circ}$.

3: (a) Find a set $X \subset \mathbb{R}^2$ with $|X| = 4$ such that $\text{Sym}(X) = \{I, F_{x+2y=5}\}$.

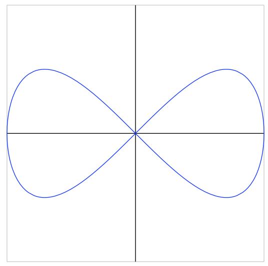
Solution: One such set X is $X = \{(0,0),(1,2),(2,4),(5,0)\}$. Another is $X = \{(-3,4),(1,2),(3,1),(5,0)\}$.

(b) Find a set $Y \subset \mathbb{R}^2$ with $|Y| = 4$ such that $\text{Sym}(Y) = \{I, R_{(1,2),\pi}\}$.

Solution: One such set is $Y = \{(0,1),(0,2),(2,2),(2,3)\}$.

(c) Find $\text{Sym}(Z)$, where $Z = \{(x,y) \in \mathbb{R}^2 \,|\, y^2 = x^2 - x^4\}$.

Solution: We draw a picture of the set Z.



From the picture we see that $\text{Sym}(Z) = D_2 = \Big\{I, R_{0,\pi}, F_{x=0}, F_{y=0}\Big\}$.

(d) Let $X = \{(x,y,z) \in \mathbb{R}^3 \,|\, xyz = 0\}$ and let G be the rotation group of X. Determine whether the rotation group of X is isomorphic to $\mathbb{Z}_n$, D_n, A_4, A_5 or to S_4.

Solution: The rotation group G permutes the 6 points $(\pm 1,0,0)$, $(0,\pm 1,0)$, $(0,0,\pm 1)$. Let $x = (1,0,0)$. Then $\big|\text{Orb}(x)\big| = 6$ and $\big|\text{Stab}(x)\big| = 4$ and so $|G| = 24$. Thus G must be isomorphic to $\mathbb{Z}_{24}$, D_{12} or S_4. Since G includes the rotations by $\pm 90°$ about each of the 3 coordinate axes, so G has at least 6 elements of order 4 (unlike $\mathbb{Z}_{24}$ and D_{12} which each have only 2 elements of order 4), and so we must have $G \cong S_4$.

4: (a) How many 8-bead necklaces (up to D_8 symmetry) can be made using 4 colours?

Solution: Let X be the set of all possible colourings, without considering the D_8 symmetry, so $|X| = 4^8$. Consider D_8 as a subgroup of $\text{Perm}(X)$. We make a table showing the value of $\big|\text{Fix}(A)\big|$ for each $A \in D_8$.

A	# of such A	$\big\|\text{Fix}(A)\big\|$
I	1	4^8
R_4	1	4^4
R_2, R_6	2	4^2
$R_1, R_3, R_5.R_7$	4	4^1
F_0, F_2, F_4, F_6	4	4^5
F_1, F_3, F_5, F_7	4	4^4

So the number of colourings, up to the D_8 symmetry, is equal to the number of orbits which is equal to

$$\tfrac{1}{16}\big(1 \cdot 4^8 + 1 \cdot 4^4 + 2 \cdot 4^2 + 4 \cdot 4^1 + 4 \cdot 4^5 + 4 \cdot 4^4\big) = 4435\,.$$

(b) How many ways (up to rotational symmetry) can the faces of a regular octahedron be coloured using 4 colours?

Solution: Let G be the rotation group of the regular octahedron. If we consider G as a subgroup of the permutations of the faces, which we label by $1, 2, \cdots 8$, then $\big|\text{Orb}(1)\big| = 8$ and $\big|\text{Stab}(1)\big| = 3$ and so we have $|G| = 24$. Now, let X be the set of all colourings, without considering the symmetry, so that $|X| = 4^8$, and consider G as a subgroup of $\text{Perm}(X)$. We make a table showing $\big|\text{Fix}(A)\big|$ for each $A \in G$.

A	#	$\big\|\text{Fix}(A)\big\|$	
the identity	1	4^8	
rotation by $\pm 120°$ about an axis through a pair of opposite faces	8	4^4	(2 groups of 3, 2 groups of 1)
rotation by $180°$ about an axis through a pair of opposite edges	6	4^4	(4 groups of 2)
rotation by $\pm 90°$ about an axis through a pair of opposite vertices	6	4^2	(2 groups of 4)
rotation by $180°$ about an axis through a pair of opposite vertices	3	4^4	(4 groups of 4)

So the number of colourings, up to the rotational symmetry, is equal to the number of orbits which is

$$\tfrac{1}{24}\big(1 \cdot 4^8 + 8 \cdot 4^4 + 6 \cdot 4^4 + 6 \cdot 4^2 + 3 \cdot 4^4\big) = 2916.$$