PMATH 336 Intro to Group Theory, Solutions to Assignment 6

1: Let $A=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, $B=\begin{pmatrix}0&i\\i&0\end{pmatrix}$ and $C=\begin{pmatrix}i&0\\0&-i\end{pmatrix}$, and let $Q_8=\langle A,B\rangle\le GL_2(\mathbb{C})$.

(a) Show that $Q_8=\{I,A,B,C,-I,-A,-B,-C\}$ and make the multiplication table for Q_8.

Solution: Let $S=\{I,A,B,C,-I,-A,-B,-C\}$. Note that $A^2=-I$ and $AB=C$ so we have

$$I=A^0\ ,\ A=A^1\ ,\ B=B^1\ ,\ C=AB\ ,\ -I=A^2\ ,\ -A=A^3\ ,\ -B=A^2B \text{ and } -C=A^2C=A^3B$$

which all lie in $\langle A,B\rangle$. Thus we have $S\subseteq\langle A,B\rangle$. Here is the multiplication table for S:

$X\backslash Y$	I	A	B	C	$-I$	$-A$	$-B$	$-C$
I	I	A	B	C	$-I$	$-A$	$-B$	$-C$
A	A	$-I$	C	$-B$	$-A$	I	$-C$	B
B	B	$-C$	$-I$	A	$-B$	C	I	$-A$
C	C	B	$-A$	$-I$	$-C$	$-B$	A	I
$-I$	$-I$	$-A$	$-B$	$-C$	I	A	B	C
$-A$	$-A$	I	$-C$	B	A	$-I$	C	B
$-B$	$-B$	C	I	$-A$	B	$-C$	$-I$	A
$-C$	$-C$	$-B$	A	I	C	B	$-A$	$-I$

The table shows that the set S is closed under multiplication and that each element in S has an inverse in S, and hence $S\le GL_2(\mathbb{C})$. Since $S\le GL_2(\mathbb{C})$ and $\{A,B\}\subseteq S$ we have $\langle A,B\rangle\subseteq S$ by the definition of $\langle A,B\rangle$.

(b) Find the number of elements of each order in Q_8.

Solution: With the help of the multiplication table, we make a table of powers, and we list the order of each element on the last row.

X	I	A	B	C	$-I$	$-A$	$-B$	$-C$
X^2	I	$-I$	$-I$	$-I$	I	$-I$	$-I$	$-I$
X^3	I	$-A$	$-B$	$-C$	$-I$	A	B	C
X^4	I	I	I	I	I	I	I	I
$\lvert X\rvert$	1	4	4	4	2	4	4	4

We see that Q_8 has 1 element of order 1, 1 element of order 2, and 6 elements of order 4.

(c) Find an abelian group which has the same number of elements of each order as $\mathbb{Z}_2\times Q_8$.

Solution: In $\mathbb{Z}_2\times Q_8$ we have

$\lvert a\rvert$	#	$\lvert X\rvert$	#	$\lvert(a,X)\rvert$	#
1	1	1	1	1	1
		2	1	2	1
		4	6	4	6
2	1	1	1	2	1
		2	1	2	1
		4	6	4	6

$\lvert(a,X)\rvert$	#
1	1
2	3
4	12

and in $\mathbb{Z}_4\times\mathbb{Z}_4$ we have

$\lvert a\rvert$	#	$\lvert b\rvert$	#	$\lvert(a,b)\rvert$	#
1	1	1	1	1	1
		2	1	2	1
		4	2	4	2
2	1	1	1	2	1
		2	1	2	1
		4	2	4	2
4	2	1	1	4	2
		2	1	4	2
		4	2	4	4

$\lvert(a,b)\rvert$	#
1	1
2	3
4	12

Thus the groups $\mathbb{Z}_2\times Q_8$ and $\mathbb{Z}_4\times\mathbb{Z}_4$ have the same number of elements of each order.

2: For each of the following groups G, find a group of the form $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_\ell}$ with $n_k \big| n_{k+1}$ for $1 \leq k < \ell$ which is isomorphic to G.

(a) $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{25} \times \mathbb{Z}_{25} \times \mathbb{Z}_7$

Solution: $G \cong (\mathbb{Z}_2 \times \mathbb{Z}_5) \times (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}) \times (\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_{25} \times \mathbb{Z}_7) \cong \mathbb{Z}_{10} \times \mathbb{Z}_{150} \times \mathbb{Z}_{2100}$.

(b) $G = U_{450}$

Solution: The powers of 2 modulo 9 are $(2^k)_{k\geq 0} = (1, 2, 4, 8, 7, 5, 1, \cdots$ so that we have $U(9) = \langle 2 \rangle \cong \mathbb{Z}^6$ and the powers of 2 modulo 25 are $(2^k)_{k\geq 0} = (1, 2, 4, 8, 16, 7, 14, 3, 6, 12, 24, 21, 17, 9, 18, 22, 19, 13, 1, \cdots)$ so that $U_{25} = \langle 2 \rangle \cong \mathbb{Z}_{20}$. Thus

$$U_{450} \cong U_2 \times U_9 \times U_{25} \cong \mathbb{Z}_1 \times \mathbb{Z}_6 \times \mathbb{Z}_{20} \cong \mathbb{Z}_6 \times \mathbb{Z}_{20} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \cong \mathbb{Z}_2 \times \mathbb{Z}_{60}.$$

(c) $G = U_{100}/\langle 49 \rangle$

Solution: Let $G = U(100)/\langle 49 \rangle$. We have $U(100) \cong U(4) \times U(25) \cong \mathbb{Z}_2 \times \mathbb{Z}_{20}$, so $|U(100)| = 40$, and $\langle 49 \rangle = \{1, 49\}$ so $|G| = 20$. So we must have $G \cong \mathbb{Z}_{20}$ or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$, since these are the only two abelian groups of order 20. The powers of 3 mod 100 are $(3^k)_{k\geq 0} = (1, 3, 9, 27, 81, 43, 29, 87, 61, 83, 49, \ldots)$, so $3^{10} \in \langle 49 \rangle$ and $|3\langle 49 \rangle| = 10$ in G. Also $3^5 = 43$ and $|43\langle 49 \rangle| = 2$ in G. Notice that we also have $|7\langle 49 \rangle| = 2$, so G contains at least 2 elements of order 2. Since $\mathbb{Z}_{20}$ only has 1 element of order 2, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$.

(d) $G = \mathbb{Z}^2/H$ where $H = \text{Span}_{\mathbb{Z}}\big\{(2, -2), (4, 2)\big\}$.

Solution: Using a picture, and some geometric intuition, we can solve this problem as follows. The distinct cosets are given by $(a, b) + H$ where the points (a, b) lie inside, and on the left edges of the parallelogram with vertices at $(0, 0)$, $(2, -2)$, $(4, 2)$ and $(6, 0)$. There are 9 points (a, b) inside, namely $(2, -1)$, $(3, -1)$, $(1, 0)$, $(2, 0)$, $(3, 0)$, $(4, 0)$, $(5, 0)$, $(3, 1)$ and $(4, 1)$, and 3 points on the edges, namely $(0, 0)$, $(1, -1)$ and $(2, 1)$. Thus we have $|G| = 9 + 3 = 12$. Also, with the help of a picture, it is easy to see that the cosets $(2, 1) + H$, $(3, 0) + H$ and $(1, -1) + H$ all have order 2. Since $\mathbb{Z}_{12}$ only has 1 element of order 2 we must have $G \cong \mathbb{Z}_2 \times \mathbb{Z}_6$.

Here is a more formal solution. Since $(4, 2) - 2(2, -2) = (0, 6)$, we have $H = \text{Span}\big\{(2, -2), (0, 6)\big\}$. Notice that for $u = (1, -1)$ and $v = (0, 1)$, the set $\{u, v\}$ is a basis for $\mathbb{Z}^2$ and the set $\{2u, 6v\}$ is a basis for H (as in Theorem 6.7). Define $\phi : \mathbb{Z}^2 \to \mathbb{Z}_2 \times \mathbb{Z}_6$ by $\phi\big(k(1, -1) + l(0, 1)\big) = (k, l)$ that is $\phi(k, l - k) = (k, l)$, or equivalently by $\phi(x, y) = (x, x + y)$. This map ϕ is clearly a surjective group homomorphism. We claim that $\text{Ker}(\phi) = H$. We have $\phi(2, -2) = (2, 0) = (0, 0)$ and $\phi(0, 6) = (0, 6) = (0, 0)$ and so $H \leq \text{Ker}(\phi)$. Conversely, let $(x, y) \in \text{Ker}(\phi)$. Then $\phi(x, y) = (0, 0) \in \mathbb{Z}_2 \times \mathbb{Z}_6$, so we have $(x, x + y) = (0, 0) \in \mathbb{Z}_2 \times \mathbb{Z}_6$. Since $x = 0 \in \mathbb{Z}_2$ we can write $x = 2s$ for some $s \in \mathbb{Z}$. Since $x + y = 0 \in \mathbb{Z}_6$ we have $y = -x = -2s \mod 6$ so we can write $y = -2s + 6t$ for some $t \in \mathbb{Z}$. Then we have $(x, y) = (2t, -2s + 6t) = s(2, -2) + t(0, 6) \in \text{Span}\big\{(2, -2), (0, 6)\big\} = H$. This shows that $\text{Ker}(\phi) \leq H$ and so we have $\text{Ker}(\phi) = H$ as claimed. Since $\phi : \mathbb{Z}^2 \to \mathbb{Z}_2 \times \mathbb{Z}_6$ is a surjective homomorphism with $\text{Ker}(\phi) = H$ we have $\mathbb{Z}^2/H \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ by the First Isomorphism Theorem.

3: (a) List all the abelian groups (up to isomorphism) of order 1200.

Solution: The abelian groups of order 1200 are

$$\begin{array}{cc}
\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_5 & \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_{25} \\
\mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_5 & \mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_{25} \\
\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_5 & \mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_3\times\mathbb{Z}_{25} \\
\mathbb{Z}_8\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_5 & \mathbb{Z}_8\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_{25} \\
\mathbb{Z}_{16}\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_5 & \mathbb{Z}_{16}\times\mathbb{Z}_3\times\mathbb{Z}_{25}
\end{array}$$

(b) Determine the number of abelian groups of order 3,200,000.

Solution: For a prime p, the abelian groups of order p^k are the groups of the form $\mathbb{Z}_{p^{k_1}}\times\mathbb{Z}_{p^{k_2}}\times\cdots\times\mathbb{Z}_{p^{k_l}}$ with $k_1\le k_2\le\cdots\le k_l$ and $\sum k_i=k$, so these groups correspond to partitions $(k_1,k_2,\cdots,k_l)$ of k. Note that 3,200,000 $=2^{10}\cdot 5^5$. The number of abelian groups of order $2^{10}\cdot 5^5$ is equal to the number of order 2^{10} multiplied by the number of order 5^5, and this is equal to the number of partitions of 10 times the number of partitions of 5. There are 7 partitions of 5, namely (5), (1,4), (2,3), (1,1,3), (1,2,2), (1,1,1,2) and (1,1,1,1,1). There are 42 partitions of 10, namely (10), (1,9) ,(2,8), (3,7), (4,6), (5,5), (1,1,8), (1,2,7), (1,3,6), (1,4,5), (2,2,6), (2,3,5), (2,4,4), (3,3,4), (1,1,1,7), (1,1,2,6), (1,1,3,5), (1,1,4,4), (1,2,2,5), (1,2,3,4), (1,3,3,3), (2,2,2,4), (2,2,3,3), (1,1,1,1,6), (1,1,1,2,5), (1,1,1,3,4), (1,1,2,2,4), (1,1,2,3,3), (1,2,2,2,3), (2,2,2,2,2), (1,1,1,1,1,5), (1,1,1,1,2,4), (1,1,1,1,3,3), (1,1,1,2,2,3), (1,1,2,2,2,2), (1,1,1,1,1,1,4), (1,1,1,1,1,2,3), (1,1,1,1,2,2,2), (1,1,1,1,1,1,1,3), (1,1,1,1,1,1,2,2), (1,1,1,1,1,1,1,1,2), (1,1. . . ,1). Thus, up to isomorphism, there are $7\cdot 42=294$ abelian groups of order 3,200,000.

(c) List every abelian group of order 12,000 which has exactly 7 elements of order 2.

Solution: We have $12,000=2^5\cdot 3^1\cdot 5^3$ so the abelian groups of order 12,000 are the groups of the form $H\times\mathbb{Z}_3\times K$ where H and K are abelian groups with $|H|=2^5=32$ and $|K|=5^3=125$. The partitions of 3 are $(1,1,1)$, $(1,2)$ and $(1,3)$ and so the we must have $K=\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_5$, $\mathbb{Z}_5\times\mathbb{Z}_{25}$ or $\mathbb{Z}_{125}$. Note the number of elements in G of order 2 is equal to the number of elements in H of order 2, and the number of elements of order 2 in the group $H=\mathbb{Z}_{2^{k_1}}\times\mathbb{Z}_{2^{k_2}}\times\cdots\times\mathbb{Z}_{p^{k_l}}$ is equal to 2^l-1 (namely the elements $(a_1,a_2,\cdots,a_l)\neq(e,e,\cdots,e)$ where for each i, either $a_i=e$ or a_i is the element in $\mathbb{Z}_{p^i}$ of order 2). Since H has 7 elements of order 2 we must have $l=3$. The partitions of 5 into 3 parts are $(1,2,2)$ and $(1,1,3)$ so we must have $H=\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_4$ or $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_8$. Thus the abelian groups of order 12,000 with exactly 7 elements of order 2 are as follows

$$\begin{array}{l}
\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_5 \\
\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_{25} \\
\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_3\times\mathbb{Z}_{125} \\
\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_8\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_5 \\
\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_8\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_{25} \\
\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_8\times\mathbb{Z}_3\times\mathbb{Z}_{125}
\end{array}$$

4: For a matrix $A \in M_{n\times n}(\mathbb{Z})$, an *elementary row operation* is a row operation of one of the three forms $R_k \leftrightarrow R_\ell$, $R_k \mapsto \pm R_k$ or $R_k \mapsto R_k + t\,R_\ell$ with $t \in \mathbb{Z}$, and an *elementary column operation* is a column operation of similar form.

Let $G = \mathbb{Z}^2 = \left\{ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \,\middle|\, x, y \in \mathbb{Z} \right\}$, let $H = \text{Span}_{\mathbb{Z}}\left\{ \left(\begin{smallmatrix} 48 \\ 66 \end{smallmatrix}\right), \left(\begin{smallmatrix} 70 \\ 98 \end{smallmatrix}\right) \right\} \le G$, and let $A = \left(\begin{smallmatrix} 48 & 70 \\ 66 & 98 \end{smallmatrix}\right) \in M_2(\mathbb{Z})$.

(a) Perform elementary row operations on A to convert it into a matrix $B = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ with $a = \gcd(48, 66)$.

Solution: We perform elementary row operations as follows:

$$\begin{pmatrix} 48 & 70 \\ 66 & 98 \end{pmatrix} R2 \mapsto R_2 - R_1 \begin{pmatrix} 48 & 70 \\ 18 & 28 \end{pmatrix} R_1 \mapsto R_1 - 2R_2 \begin{pmatrix} 12 & 14 \\ 18 & 28 \end{pmatrix} R_2 \mapsto R_2 - R_1 \begin{pmatrix} 12 & 14 \\ 6 & 14 \end{pmatrix}$$
$$R_1 \mapsto R_1 - 2R_2 \begin{pmatrix} 0 & -14 \\ 6 & 14 \end{pmatrix} R_1 \leftrightarrow R_2 \begin{pmatrix} 6 & 14 \\ 0 & -14 \end{pmatrix}$$

(b) Perform elementary column operations on B to convert it to the form $C = \left(\begin{smallmatrix} k & 0 \\ \ell & m \end{smallmatrix}\right)$ with $k = \gcd(a, b)$.

Solution: We perform column operations as follows

$$\begin{pmatrix} 6 & 14 \\ 0 & -14 \end{pmatrix} C_2 \mapsto C_2 - 2C_1 \begin{pmatrix} 6 & 2 \\ 0 & -14 \end{pmatrix} C_1 \mapsto C1 - 3C_2 \begin{pmatrix} 0 & 2 \\ 42 & -14 \end{pmatrix} C_1 \leftrightarrow C_2 \begin{pmatrix} 2 & 0 \\ -14 & 42 \end{pmatrix}$$

(c) Perform elementary row and column operations on C to convert it to diagonal form $D = \left(\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right)$ with $r \,\middle|\, s$.

Solution: We perform the single operation

$$\begin{pmatrix} 2 & 0 \\ -14 & 42 \end{pmatrix} R_2 \mapsto R_2 + 7R_1 \begin{pmatrix} 2 & 0 \\ 0 & 42 \end{pmatrix}$$

(d) Use your sequence of elementary row and column operations from Parts (a), (b) and (c) to find a basis $\{u, v\}$ for $G = \mathbb{Z}^2$ such that $\{ru, sv\}$ is a basis for H.

Solution: The column operations can be used to change our basis for H:

$$\begin{pmatrix} 48 & 70 \\ 66 & 98 \end{pmatrix} C_2 \mapsto C_2 - 2C_1 \begin{pmatrix} 48 & -26 \\ 66 & -34 \end{pmatrix} C_1 \mapsto C_1 - 3C_2 \begin{pmatrix} 126 & -26 \\ 168 & 34 \end{pmatrix} C_1 \leftrightarrow C_2 \begin{pmatrix} -26 & 126 \\ -34 & 168 \end{pmatrix}$$

and the row operations can be used to change from the standard basis in $\mathbb{Z}^2$: performing the inverse row operations in reverse order gives

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R_2 \mapsto R_2 - 7R_1 \begin{pmatrix} 1 & 0 \\ -7 & 1 \end{pmatrix} R_1 \leftrightarrow R_2 \begin{pmatrix} -7 & 1 \\ 1 & 0 \end{pmatrix} R_1 \mapsto R_1 + 2R_2 \begin{pmatrix} -5 & 1 \\ 1 & 0 \end{pmatrix}$$
$$R_2 \mapsto R_2 + R_1 \begin{pmatrix} -5 & 1 \\ -4 & 1 \end{pmatrix} R_1 \mapsto R_1 + 2R_2 \begin{pmatrix} -13 & 3 \\ -4 & 1 \end{pmatrix} R_2 \mapsto R_2 + R_1 \begin{pmatrix} -13 & 3 \\ -17 & 4 \end{pmatrix}$$

Thus we can take $u = \begin{pmatrix} -13 \\ -17 \end{pmatrix}$ and $v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ with $r = 2$ and $s = 42$.