PMATH 336 Intro to Group Theory, Assignment 6 Due Tues July 30, 11:00 pm

1: Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $C = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, and let $Q_8 = \langle A, B \rangle \leq GL_2(\mathbb{C})$.

(a) Show that $Q_8 = \{I, A, B, C, -I, -A, -B, -C\}$ and make the multiplication table for Q_8.

(b) Find the number of elements of each order in Q_8.

(c) Find an abelian group which has the same number of elements of each order as $\mathbb{Z}_2 \times Q_8$.

2: For each of the following groups G, find a group of the form $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_\ell}$ with $n_k \big| n_{k+1}$ for $1 \leq k < \ell$ which is isomorphic to G.

(a) $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{25} \times \mathbb{Z}_{25} \times \mathbb{Z}_7$

(b) $G = U_{450}$

(c) $G = U_{100} / \langle 49 \rangle$

(d) $G = \mathbb{Z}^2 / H$ where $H = \text{Span}_{\mathbb{Z}}\{(2, -2), (4, 2)\}$.

3: (a) List all the abelian groups (up to isomorphism) of order 1200.

(b) Determine the number of abelian groups of order 3,200,000.

(c) List every abelian group of order 12,000 which has exactly 7 elements of order 2.

4: For a matrix $A \in M_{n \times n}(\mathbb{Z})$, an *elementary row operation* is a row operation of one of the three forms $R_k \leftrightarrow R_\ell$, $R_k \mapsto \pm R_k$ or $R_k \mapsto R_k + t\, R_\ell$ with $t \in \mathbb{Z}$, and an *elementary column operation* is a column operation of similar form.

Let $G = \mathbb{Z}^2 = \left\{ \binom{x}{y} \,\middle|\, x, y \in \mathbb{Z} \right\}$, let $H = \text{Span}_{\mathbb{Z}}\left\{ \binom{48}{66}, \binom{70}{98} \right\} \leq G$, and let $A = \begin{pmatrix} 48 & 70 \\ 66 & 98 \end{pmatrix} \in M_2(\mathbb{Z})$.

(a) Perform elementary row operations on A to convert it into a matrix $B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a = \gcd(48, 66)$.

(b) Perform elementary column operations on B to convert it to the form $C = \begin{pmatrix} k & 0 \\ \ell & m \end{pmatrix}$ with $k = \gcd(a, b)$.

(c) Perform elementary row and column operations on C to convert it to diagonal form $D = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$ with $r \big| s$.

(d) Use your sequence of elementary row and column operations from Parts (a), (b) and (c) to find a basis $\{u, v\}$ for $G = \mathbb{Z}^2$ such that $\{ru, sv\}$ is a basis for H.