PMATH 336 Intro to Group Theory, Solutions to Assignment 5

1: (a) For each of the two quotient groups $U_{16}/\langle 7\rangle$ and $U_{16}/\langle 9\rangle$, list all elements in each coset, determine the multiplication tables, and determine whether the group is isomorphic to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Solution: We have $U_{16} = \{1,3,5,7,9,11,13,15\}$, $\langle 7\rangle = \{1,7\}$ and $U_{16}/\langle 7\rangle = \big\{\{1,7\},\{3,5\},\{9,15\},\{11,15\}\big\}$, and $\langle 9\rangle = \{1,9\}$ and $U_{16}/\langle 9\rangle = \big\{\{1,9\},\{3,11\},\{5,13\},\{7,13\}\big\}$. Here are the multiplication tables:

	$\{1,7\}$	$\{3,5\}$	$\{9,15\}$	$\{11,13\}$
$\{1,7\}$	$\{1,7\}$	$\{3,5\}$	$\{9,15\}$	$\{11,13\}$
$\{3,5\}$	$\{3,5\}$	$\{9,15\}$	$\{11,13\}$	$\{1,7\}$
$\{9,15\}$	$\{9,15\}$	$\{11,13\}$	$\{1,7\}$	$\{3,5\}$
$\{11,13\}$	$\{11,13\}$	$\{1,7\}$	$\{3,5\}$	$\{9,15\}$

	$\{1,9\}$	$\{3,11\}$	$\{5,13\}$	$\{7,15\}$
$\{1,9\}$	$\{1,9\}$	$\{3,11\}$	$\{5,13\}$	$\{7,15\}$
$\{3,11\}$	$\{3,11\}$	$\{1,9\}$	$\{7,15\}$	$\{5,13\}$
$\{5,13\}$	$\{5,13\}$	$\{7,15\}$	$\{1,9\}$	$\{3,11\}$
$\{7,15\}$	$\{7,15\}$	$\{15,13\}$	$\{3,11\}$	$\{1,9\}$

From the multiplication tables, we see that $U_{16}/\langle 7\rangle \cong \mathbb{Z}_4$ and $U_{16}/\langle 9\rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(b) List all the elements in each left coset of $H = \langle F_0\rangle$ in D_4 and find all the conjugate subgroups, that is find all subgroups of D_4 of the form aHa^{-1} for some $a \in D_4$.

Solution: We have $D_4 = \{I, R_!, R_2, R_3, F_0, F_1, F_2, F_3\}$ and $\langle F_0\rangle = \{I, F_0\}$. The cosets are $I\,H = \{I, F_0\}$, $R_1H = \{R_1, F_1\}$, $R_2H = \{R_2, F_2\}$, $R_3H = \{R_3, F_3\}$, $F_0H = \{F_0, I\}$, $F_1H = \{F_1, R_1\}$, $F_2H = \{F_2, R_2\}$ and $F_3H = \{F_3, R_3\}$.

The subgroups of the form aHa^{-1} with $a \in D_4$ are $I\,H\,I = H$, $R_1H\,R_3 = \{I, F_2\}$, $R_2H\,R_2 = \{I, F_0\}$, $R_3H\,R_1 = \{I, F_2\}$, $F_0H\,F_0 = \{I, F_0\}$, $F_1H\,F_1 = \{I, F_0\}$, $F_2H\,F_2 = \{I, F_0\}$ and $F_3H\,F_3 = \{I, F_0\}$. Thus there are two distinct such subgroups, namely $\langle F_0\rangle = \{I, F_0\}$ and $\langle F_2\rangle = \{I, F_2\}$.

2: (a) Let $H = \big\{(1),(134),(143),(13),(14),(34)\big\} \leq S_4$. List all of the elements in each left coset of H in S_4 and determine whether $H \trianglelefteq S_4$.

Solution: The left cosets are

$$\begin{aligned}(1)H &= \big\{(1),(134),(143),(13),(14),(34)\big\}\\ (12)H &= \big\{(12),(1342),(1432),(132),(142),(12)(34)\big\}\\ (23)H &= \big\{(23),(1234),(1423),(123),(14)(23),(234)\big\}\\ (24)H &= \big\{(24),(1324),(1243),(13)(24),(124),(243)\big\}\end{aligned}$$

and the right coset containing (12) is

$$H(12) = \big\{(12),(1234),(1243),(123),(124),(12)(34)\big\}.$$

Since $(12)H \neq H(12)$, we see that H is not a normal subgroup of G.

(b) Let $N = \{(1),(12)(34),(13)(24),(14)(23)\} \leq S_4$. Show that $N \trianglelefteq S_4$ and determine whether S_4/N is isomorphic to $\mathbb{Z}_6$ or to D_3.

Solution: The left cosets are

$$\begin{aligned}(1)N &= N,\\ (12)N &= \{(12),(34),(1324),(1423)\},\\ (13)N &= \{(13),(1234),(24),(1432)\},\\ (14)N &= \{(14),(1243),(1342),(23)\},\\ (123)N &= \{(123),(134),(243),(142)\} \text{ and}\\ (124)N &= \{(124),(143),(132),(234)\}\end{aligned}$$

and the right cosets are

$$\begin{aligned}N(1) &= N,\\ N(12) &= \{(12),(34),(1423),(1324)\},\\ N(13) &= \{(13),(1432),(24),(1234)\},\\ N(14) &= \{(14),(1342),(1243),(23)\},\\ N(123) &= \{(123),(243),(142),(134)\} \text{ and}\\ N(124) &= \{(124),(234),(143),(132)\}.\end{aligned}$$

Since the left cosets are equal to the right cosets, N is normal.

In S_4/N we have $\big((12)N\big)^2 = \big((13)N\big)^2 = \big((14)N\big)^2 = N$, so S_4/N has (at least) 3 elements of order 2 while $\mathbb{Z}_6$ has only 2 elements of order 2, so $S_4/N \cong D_3$.

3: (a) (The Orbit/Stabilizer Theorem) Let A be a nonempty set and let G be a finite subgroup of $\text{Perm}(A)$. For $a \in A$, the **orbit** of a is the set $\text{Orb}(a) = \big\{\sigma(a) \,\big|\, \sigma \in G\big\} \subseteq A$, and the **stabilizer** of a is the set $\text{Stab}(a) = \big\{\sigma \in G \,\big|\, \sigma(a) = a\big\}$. Show that for all $a \in A$, we have $\text{Stab}(a) \leq G$ and $|G| = \big|\text{Orb}(a)\big|\,\big|\text{Stab}(a)\big|$.

Solution: We note that $\text{Stab}(a)$ is a subgroup of G by the Finite Subgroup Test because the identity element is the identity function I which satisfies $I(a) = a$ so that $I \in \text{Stab}(a)$, and because given $\sigma, \tau \in \text{Stab}(a)$ so that $\sigma(a) = a$ and $\tau(a) = a$, we have $(\sigma\tau)(a) = \sigma(\tau(a)) = \sigma(a) = a$ so that $\sigma\tau \in \text{Stab}(a)$.

Define $F : G/\text{Stab}(a) \to \text{Orb}(a)$ by $F\big(\sigma\,\text{Stab}(a)\big) = \sigma(a)$, where $\sigma \in G$. Note that F is well-defined because for $\sigma, \tau \in G$, if $\sigma\,\text{Stab}(a) = \tau\,\text{Stab}(a)$ then $\tau^{-1}\sigma \in \text{Stab}(a)$ so that $\tau^{-1}\sigma(a) = a$ and hence $\sigma(a) = \tau\tau^{-1}\sigma(a) = \tau(\tau^{-1}\sigma(a)) = \tau(a)$. The map F is clearly surjective, and F is also injective because, given $\sigma, \tau \in G$, if $F\big(\sigma\,\text{Stab}(a)\big) = F\big(\tau\,\text{Stab}(a)\big)$ then we have $\sigma(a) = \tau(a)$ and hence $\tau^{-1}\sigma(a) = a$ so that $\sigma\,\text{Stab}(a) = \tau\,\text{Stab}(a)$. Since F is bijective, we have $\big|G/\text{Stab}(a)\big| = \big|\text{Orb}(a)\big|$. By Lagrange's Theorem, it follows that $|G| = \big|G/\text{Stab}(a)\big|\,\big|\text{Stab}(a)\big| = \big|\text{Orb}(a)\big|\,\big|\text{Stab}(a)\big|$.

(b) Let $G = \big\{(1), (13)(46), (13)(25), (14)(36), (16)(34), (25)(46), (1436)(25), (1634)(25)\big\} \leq \text{Perm}\{1, 2, \cdots, n\}$. Find $\text{Orb}(1)$, $\text{Stab}(1)$, $\text{Orb}(2)$ and $\text{Stab}(2)$.

Solution: From the definition of $\text{Orb}(a)$ and $\text{Stab}(a)$, we have

$$\begin{aligned}
\text{Orb}(1) &= \{1, 3, 4, 6\} \\
\text{Stab}(1) &= \big\{(1), (25)(46)\big\} \\
\text{Orb}(2) &= \{2, 5\} \\
\text{Stab}(2) &= \big\{(1), (13)(46), (14)(36), (16)(34)\big\}
\end{aligned}$$

(c) Let $G = GL_3(\mathbb{Z}_2) \leq \text{Perm}({\mathbb{Z}_2}^3)$ and let $a = e_1 = (1, 0, 0)^T \in {\mathbb{Z}_2}^3$. Find $\big|\text{Orb}(a)\big|$ and $\big|\text{Stab}(a)\big|$.

Solution: For $A \in GL_3(\mathbb{Z}_2)$ note that Ae_1 is the first column of A. So the orbit of e_1 is the set of all possible leading columns of the matrices in $GL(3, \mathbb{Z}_2)$. Since the first column can be anything other than the zero vector, there are $2^3 - 1 = 7$ possible leading columns. Thus $\big|\text{Orb}(e_1)\big| = 7$. Since we have $\big|GL(2, \mathbb{Z}_3)\big| = (8-1)(8-2)(8-4) = 168$, the Orbit/Stabilizer theorem implies that $\big|\text{Stab}(e_1)\big| = 168/7 = 24$.

4: (a) Find all the homomorphisms $\phi : \mathbb{Z}_4 \to \mathbb{C}^*$. For each one, describe its kernel and image.

Solution: The homomorphisms are the maps $\phi_a : \mathbb{Z}_4 \to \mathbb{C}^*$ given by $\phi_a(k) = a^k$, where $a \in \mathbb{C}^*$ with $a^4 = 1$, that is $a = \pm 1, \pm i$. Explicitly, the maps ϕ_a are given by

k	0	1	2	3
$\phi_1(k)$	1	1	1	1
$\phi_{-1}(k)$	1	-1	1	-1
$\phi_i(k)$	1	i	-1	$-i$
$\phi_{-i}(k)$	1	$-i$	-1	i

So $\text{Ker}(\phi_1) = \mathbb{Z}_4$, $\text{Image}(\phi_1) = \{1\}$, $\text{Ker}(\phi_{-1}) = \{0, 2\}$, $\text{Image}(\phi_{-1}) = \{\pm 1\}$, $\text{Ker}(\phi_i) = \text{Ker}(\phi_{-i}) = \{0\}$, and $\text{Image}(\phi_i) = \text{Image}(\phi_{-i}) = \{\pm 1, \pm i\}$.

(b) Show that $\mathbb{S}^1/C_n \cong \mathbb{S}^1$ where $\mathbb{S}^1 = \{z \in \mathbb{C}^* \big| |z| = 1\}$ and $C_n = \{z \in \mathbb{C}^* \big| z^n = 1\}$.

Solution: Define $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ by $\phi(e^{i\theta}) = e^{in\theta}$, or equivalently by $\phi(z) = z^n$. Note that ϕ is a homomorphism because $\phi(zw) = (zw)^n = z^n w^n = \phi(z)\phi(w)$. Note that ϕ is surjective because given $w = e^{i\theta} \in \mathbb{S}^1$ we can take $z = e^{i\theta/n}$ to get $\phi(z) = w$. Finally note that $z \in \text{Ker}(\phi) \iff \phi(z) = 1 \iff z^n = 1 \iff z \in C_n$ and so $\text{Ker}(\phi) = C_n$. By the First Isomorphism Theorem, we have $\mathbb{S}^1/C_n \cong \mathbb{S}^1$.

(c) Let H be the spiral $H = \{r\, e^{i\theta} \in \mathbb{C}^* \big| r = e^{\theta/\pi}\} \leq \mathbb{C}^*$. Show that $\mathbb{C}^*/H \cong \mathbb{S}^1$.

Solution: Note that for $r, \theta \in \mathbb{R}$ we have $r = e^{\theta/\pi} \iff \theta = \pi \ln r$. Define $\phi : \mathbb{C}^* \to \mathbb{S}^1$ by

$$\phi(re^{i\theta}) = e^{i(\theta - \pi \ln r)} = e^{i\pi} e^{-\pi \ln r}\ , \text{ or equivalently } \phi(z) = \tfrac{z}{|z|}\, e^{-i\pi \ln |z|}\,.$$

Note that ϕ is a group homomorphism since for $z, w \in \mathbb{C}^*$ we have

$$\phi(zw) = \tfrac{zw}{|zw|}\, e^{-i\pi \ln |zw|} = \tfrac{z}{|z|}\, \tfrac{w}{|w|}\, e^{-i\pi(\ln|z| + \ln w)} = \tfrac{z}{|z|}\, e^{-i\pi \ln|z|} \cdot \tfrac{w}{|w|}\, e^{-i\pi \ln|w|} = \phi(z)\phi(w)\,.$$

Also note that ϕ is surjective since when $|z| = 1$ we have $\phi(z) = z\, e^{-i\pi \ln 1} = z\, e^0 = z$. Finally, we claim that $\text{Ker}(\phi) = H$. Let $z \in H$, say $z = re^{i\theta}$ with $r = e^{\theta/\pi}$, so $\theta = \pi \ln r$. Then

$$\phi(z) = \phi(re^{i\theta}) = e^{i(\theta - \pi \ln r)} = e^{i(\theta - \theta)} = e^0 = 1$$

so we have $z \in \text{Ker}(\phi)$. Conversely, suppose that $z \in \text{Ker}(\phi)$, say $z = r\, e^{i\alpha}$ where $r, \alpha \in \mathbb{R}$ with $r > 0$. Then

$$1 = \phi(z) = \phi(r\, e^{i\alpha}) = e^{i(\alpha - \pi \ln r)}$$

so we can choose $k \in \mathbb{Z}$ so that $\alpha - \pi \ln r = 2\pi k$. Let $\theta = \alpha - 2\pi k$. Then we have $e^{i\alpha} = e^{i\theta}$ so that $z = re^{i\alpha} = re^{i\theta}$, and we have $\theta = \pi \ln r$ so that $z = re^{i\theta} \in H$. Thus $\text{Ker}(\phi) = H$, as claimed. By the First Isomorphism Theorem, $H \trianglelefteq \mathbb{C}^*$ and $\mathbb{C}^*/H \cong \mathbb{S}^1$.