AMATH/PMATH 331 Real Analysis, Solutions to the Problems for Chapter 8

1: Let $f:\mathbf{R}\to\mathbf{R}$ be the 2π-periodic function with $f(x)=x^3-\pi^2x$ for $-\pi\leq x\leq\pi$.

(a) Find the coefficients of the (real) Fourier series for f.

Solution: Since $f(x)$ is odd we have $a_0=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\,dx=0$ and $a_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\cos nx\ dx=0$ and $b_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\sin nx\ dx=\dfrac{2}{\pi}\displaystyle\int_0^{\pi}(x^3-\pi^2x)\sin nx\ dx$. Integration by Parts gives

$$\int_0^\pi x\,\sin nx\ dx=\Big[-\tfrac{1}{n}\ x\,\cos nx\Big]_0^\pi+\int_0^\pi\tfrac{1}{n}\cos nx\ dx=-\tfrac{1}{n}\,\pi\,\cos n\pi=-\tfrac{(-1)^n\pi}{n}.$$

and

$$\begin{aligned}\int_0^\pi x^3\sin nx\ dx&=\Big[-\tfrac{1}{n}x^3\cos nx\Big]_0^\pi+\int_0^\pi\tfrac{3}{n}x^2\cos nx\ dx\\&=-\tfrac{(-1)^n\pi^3}{n}+\Big[\tfrac{3}{n^2}x^2\sin nx\Big]_0^\pi-\int_0^\pi\tfrac{6}{n^2}x\sin nx\ dx\\&=-\tfrac{(-1)^n\pi^3}{n}+0+\tfrac{6}{n^2}\,\tfrac{(-1)^n\pi}{n}=(-1)^n\Big(\tfrac{6\pi}{n^3}-\tfrac{\pi^3}{n}\Big)\end{aligned}$$

and so

$$b_n=\frac{2}{\pi}\int_0^\pi(x^3-\pi^2x)\sin nx\ dx=\tfrac{2}{\pi}\Big((-1)^n\big(\tfrac{6\pi}{n^3}-\tfrac{\pi^3}{n}\big)+(-1)^n\tfrac{\pi^3}{n}\Big)=\tfrac{(-1)^n12}{n^3}.$$

(b) Show that $s_m(f)\to f$ uniformly on $\mathbf{R}$.

Solution: Since $s_m(f)(x)=\sum\limits_{n=1}^{m}\frac{(-1)^n12}{n^3}\sin nx$ and $\left|\frac{(-1)^n12}{n^3}\sin nx\right|\leq\frac{12}{n^3}$, it follows from the Weierstrass M Test that $\{s_m(f)(x)\}$ converges uniformly on $\mathbf{R}$ (to some function g), and by Fejér's Theorem we have $\lim\limits_{\ell\to\infty}s_m(f)(x)=\lim\limits_{\ell\to\infty}\sigma_\ell(f)(x)=f(x)$ for all $x\in\mathbf{R}$.

(c) By evaluating at $x=\frac{\pi}{2}$, evaluate $\sum\limits_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}$.

Solution: Since $f(x)=x^3-\pi^2x$ for $-\pi\leq x\leq\pi$, we have $f\big(\frac{\pi}{2}\big)=\big(\frac{\pi}{2}\big)^3-\pi^2\big(\frac{\pi}{2}\big)=-\frac{3\pi^3}{8}$. On the other hand, since $f(x)=\sum\limits_{n=1}^{\infty}\frac{(-1)^n12}{n^3}\sin nx$, and since when $n=2k$ we have $\sin\frac{n\pi}{2}=0$ and when $n=2k+1$ we have $\sin\frac{n\pi}{2}=(-1)^k$, we have $f\big(\frac{\pi}{2}\big)=\sum\limits_{n=1}^{\infty}\frac{(-1)^n12}{n^3}\sin\frac{n\pi}{2}=\sum\limits_{k=0}^{\infty}\frac{(-1)^{2k+1}12}{(2k+1)^3}(-1)^k=-\sum\limits_{k=0}^{\infty}\frac{(-1)^k12}{(2k+1)^3}$. Thus

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}=-\tfrac{1}{12}\,f\big(\tfrac{\pi}{2}\big)=\tfrac{1}{12}\cdot\tfrac{3\pi^3}{8}=\tfrac{\pi^3}{32}.$$

2: Let $f:\mathbf{R}\to\mathbf{R}$ be a 2π-periodic function whose restriction to $[-\pi,\pi]$ is continuous.

(a) Use Integration by Parts to show that if f is $\mathcal{C}^1$ (meaning that the derivative f' exists and is continuous) then $\left|c_n(f)\right|\leq\frac{M}{|n|}$ for all $n\in\mathbf{Z}$ where $M=\|f'\|_\infty=\max\limits_{-\pi\leq t\leq\pi}\left|f'(t)\right|$.

Solution: Suppose that $f\in\mathcal{C}^1$ and let $M=\max\limits_{-\pi\leq t\leq\pi}\left|f'(t)\right|$. Integration by Parts gives

$$c_n(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)e^{-int}\,dt=\frac{1}{2\pi}\left(\Big[\frac{i}{n}\,f(t)e^{-int}\Big]_{-\pi}^{\pi}-\frac{i}{n}\int_{-\pi}^{\pi}f'(t)e^{-int}dt\right)=\frac{-i}{2\pi n}\int_{-\pi}^{\pi}f'(t)e^{-int}dt.$$

Thus we have $c_n(f)=\frac{-i}{2\pi n}c_n(f')$, and

$$\left|c_n(f)\right|\leq\frac{1}{2\pi|n|}\int_{-\pi}^{\pi}\left|f'(t)\right|dt\leq\frac{1}{2\pi|n|}\,2\pi M=\frac{M}{|n|}.$$

(b) Use induction to show that if f in $\mathcal{C}^k$ (meaning that the k^{th} derivative of f exists and is continuous) then $\left|c_n(f)\right|\leq\frac{M}{(2\pi)^{k-1}|n|^k}$ for all $n\in\mathbf{Z}$ where $M=\left\|f^{(k)}(x)\right\|_\infty=\max\limits_{-\pi\leq x\leq\pi}\left|f^{(k)}(x)\right|$.

Solution: Let $f\in\mathcal{C}^k$ and let $M=\max\limits_{-\pi\leq t\leq\pi}\left|f^{(k)}(t)\right|$. In Part (a) we showed that $c_n(f)=\frac{-i}{2\pi n}\,c_n(f')$ and it follows, by induction, that $c_n(f)=\left(\frac{-i}{2\pi n}\right)^k c_n\big(f^{(k)}\big)$, hence

$$\left|c_n(f)\right|=\left|\frac{1}{(2\pi)^k|n|^k}\int_{-\pi}^{\pi}f^{(k)}(t)e^{-int}\,dt\right|\leq\frac{1}{(2\pi)^k|n|^k}\int_{-\pi}^{\pi}\left|f^{(k)}(t)\right|dt\leq\frac{1}{(2\pi)^k|n|^k}\,2\pi M=\frac{M}{(2\pi)^{k-1}|n|^k}.$$

(c) Show that if $f\in\mathcal{C}^2$ then $s_m(f)\to f$ uniformly on $\mathbf{R}$.

Solution: Let $f\in\mathcal{C}^2$ and let $M=\max\limits_{-\pi\leq t\leq\pi}\left|f''(t)\right|$. By Part (b) we have $|c_n(f)|\leq\frac{M}{2\pi n^2}$ for all $n\in\mathbf{Z}$. Since $s_m(f)(x)=\sum\limits_{n=-m}^{m}c_n(f)e^{inx}$ and $\left|c_n(f)e^{inx}\right|=\left|c_n(f)\right|\leq\frac{M}{2\pi n^2}$, the Weierstrass M Test shows that the sequence $\left\{s_m(f)(x)\right\}$ converges uniformly on $\mathbf{R}$ $\big($to some function $g(x)\big)$. By Fejér's Theorem, we have $\lim\limits_{m\to\infty}s_m(f)(x)=\lim\limits_{\ell\to\infty}\sigma_\ell(f)(x)=f(x)$ for all $x\in\mathbf{R}$.

3: Let $f \in \mathcal{R}(T)$, let $(c_n)_{n\geq 0}$ and $(d_n)_{n\geq 1}$ be sequences in $\mathbf{R}$ and let $p_m(x) = c_0 + \sum_{n=1}^{m} c_n \cos nx + \sum_{n=1}^{m} d_n \sin nx$.

(a) Show that if $p_m \to f$ in $\big(\mathcal{R}(T), \|\ \|_1\big)$ then $c_0 = a_0(f)$ and $c_n = a_n(f)$ and $d_n = b_n(f)$ for all $n \in \mathbf{Z}^+$. for all $n \in \mathbf{Z}$.

Solution: Suppose $p_m \to f$ in $\big(\mathcal{R}(T), \|\ \|_1\big)$. Let $\epsilon > 0$. Choose $\ell \in \mathbf{Z}^+$ so that $m \geq \ell \Longrightarrow \|p_m - f\|_1 < 2\pi\epsilon$. Then for $m \geq \ell$ we have

$$\begin{aligned}
\big|c_0 - a_n(f)\big| &= \left|\tfrac{1}{2\pi}\int_{-\pi}^{\pi} p_m(x) - f(x)\ dx\right| \leq \tfrac{1}{2\pi}\int_{-\pi}^{\pi} \big|p_m(x) - f(x)\big|\,dx = \tfrac{1}{2\pi}\|p_m - f\|_1 < \epsilon, \\
\big|c_n - a_n(f)\big| &= \left|\tfrac{1}{\pi}\int_{-\pi}^{\pi} \big(p_m(x) - f(x)\big)\cos nx\ dx\right| \leq \tfrac{1}{\pi}\int_{-\pi}^{\pi} \big|p_m(x) - f(x)\big|\,dx = \tfrac{1}{\pi}\|p_m - f\|_1 < 2\epsilon, \\
\big|d_n - a_n(f)\big| &= \left|\tfrac{1}{\pi}\int_{-\pi}^{\pi} \big(p_m(x) - f(x)\big)\sin nx\ dx\right| \leq \tfrac{1}{\pi}\int_{-\pi}^{\pi} \big|p_m(x) - f(x)\big|\,dx = \tfrac{1}{\pi}\|p_m - f\|_1 < 2\epsilon.
\end{aligned}$$

Since $|c_0 - a_0(f)| < \epsilon$, $|c_n - a_n(f)| < 2\epsilon$ and $|d_n - b_n(f)| < 2\epsilon$ for all $\epsilon > 0$ it follows that $c_0 = a_0(f)$, $c_n = a_n(f)$ and $d_n = b_n(f)$.

(b) Show that if $p_m \to f$ in $\big(\mathcal{R}(T), \|\ \|_\infty\big)$ then $c_0 = a_0(f)$ and $c_n = a_n(f)$ and $d_n = b_n(f)$ for all $n \in \mathbf{Z}^+$.

Solution: Suppose $p_m \to f$ in $\big(\mathcal{R}(T), \|\ \|_\infty\big)$, that is $p_m \to f$ uniformly on $[-\pi, \pi]$. We remark that since $p_m \to f$ uniformly on $[-\pi, \pi]$, it follows that f is continuous (but we do not use the fact that f is continuous in our solution). Let $\epsilon > 0$. Choose $\ell \in \mathbf{Z}^+$ so that $m \geq \ell \Longrightarrow \|p_m - f\|_\infty < \epsilon$. Then for $m \geq \ell$ we have

$$\begin{aligned}
\big|c_0 - a_n(f)\big| &= \left|\tfrac{1}{2\pi}\int_{-\pi}^{\pi} p_m(x) - f(x)\ dx\right| \leq \tfrac{1}{2\pi}\int_{-\pi}^{\pi} \big|p_m(x) - f(x)\big|\,dx \\
&\leq \tfrac{1}{2\pi}\int_{-\pi}^{\pi} \|p_m - f\|_\infty = \|p_m - f\|_\infty < \epsilon, \\
\big|c_n - a_n(f)\big| &= \left|\tfrac{1}{\pi}\int_{-\pi}^{\pi} \big(p_m(x) - f(x)\big)\cos nx\ dx\right| \leq \tfrac{1}{\pi}\int_{-\pi}^{\pi} \big|p_m(x) - f(x)\big|\,dx \\
&\leq \tfrac{1}{\pi}\int_{-\pi}^{\pi} \|p_m - f\|_\infty\,dx < 2\epsilon,
\end{aligned}$$

and similarly $\big|d_n - b_n(f)\big| < 2\epsilon$. Since $|c_0 - a_0(f)| < \epsilon$, $|c_n - a_n(f)| < 2\epsilon$ and $|d_n - b_n(f)| < 2\epsilon$ for all $\epsilon > 0$ it follows that $c_0 = a_0(f)$, $c_n = a_n(f)$ and $d_n = b_n(f)$.

(c) Show that if $p_m \to f$ in $\big(\mathcal{R}(T), \|\ \|_2\big)$ then $c_0 = a_0(f)$ and $c_n = a_n(f)$ and $d_n = b_n(f)$ for all $n \in \mathbf{Z}^+$.

Solution: Suppose $p_m \to f$ in $\big(\mathcal{R}(T), \|\ \|_2\big)$. Let $\epsilon > 0$, Choose $\ell \in \mathbf{Z}^+$ so that $m \leq \ell \Longrightarrow \|p_m - f\|_2 < \frac{1}{\sqrt{2\pi}}\,\epsilon$. By the Cauchy-Schwarz Inequality, for $m \geq \ell$ we have

$$\|p_m - f\|_1 = \int_{-\pi}^{\pi} \big|p_m(x) - f(x)\big|\,dx = \big\langle 1, |p_m - f|\big\rangle \leq \|1\|_2 \big\||p_m - f|\big\|_2 = \sqrt{2\pi}\,\|p_m - f\|_2 < \epsilon.$$

Thus $p_m \to f$ in $\big(\mathcal{R}(T), \|\ \|_1\big)$, so $c_0 = a_0(f)$ and $c_n = a_n(f)$ and $d_n = b_n(f)$ for all $n \in \mathbf{Z}^+$ by Part (a).

4: Let $f:\mathbf{R}\to\mathbf{R}$ be the 2π-periodic function with $f(x)=\begin{cases} 1 \text{ if } 0<x<\pi, \\ -1 \text{ if } -\pi<x<0, \\ 0 \text{ if } x=0,\pm\pi. \end{cases}$

(a) Find the coefficients of the (real) Fourier series for f.

Solution: Since $f(x)$ is odd we have $a_0=0$ and $a_n=0$ for all $n\in\mathbf{Z}^+$ and we have

$$b_n=\tfrac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx=\tfrac{2}{\pi}\int_0^\pi \sin nx\,dx=\tfrac{2}{\pi}\Big[-\tfrac{1}{n}\cos nx\Big]_0^\pi=-\tfrac{2}{\pi n}\big((-1)^n-1\big)=\begin{cases}\frac{4}{n\pi} \text{ if } n \text{ is odd.}\\ 0 \text{ if } n \text{ is even.}\end{cases}$$

(b) By recognizing $s_{2m}(f)\big(\frac{\pi}{2m}\big)$ as a Riemann sum, show that $\lim\limits_{m\to\infty}s_{2m}(f)\big(\frac{\pi}{2m}\big)=\dfrac{2}{\pi}\displaystyle\int_0^\pi\frac{\sin x}{x}\,dx$.

Solution: When we partition the interval $[0,\pi]$ into m equal-sized subintervals, the endpoints of the subintervals are $x_k=\frac{\pi k}{m}$ and the midpoints of the subintervals are $m_k=\frac{x_k+x_{k-1}}{2}=\frac{(2k-1)\pi}{2m}$. The Riemann sum for $\displaystyle\int_0^\pi\frac{\sin x}{x}\,dx$ using the midpoints of this partition is

$$R_m=\sum_{k=1}^m\frac{\sin m_k}{m_k}(x_k-x_{k-1})=\sum_{k=1}^m\frac{\sin\frac{(2k-1)\pi}{2m}}{\frac{(2k-1)\pi}{2m}}\cdot\frac{\pi}{m}=2\sum_{k=1}^m\frac{\sin\frac{(2k-1)\pi}{2m}}{(2k-1)}$$

By Part (a) we have

$$s_{2m}(f)(x)=s_{2m-1}(f)(x)=\sum_{\substack{n\text{ odd}\\1\le n\le 2m}}\tfrac{4}{n\pi}\sin nx=\tfrac{4}{\pi}\sum_{k=1}^m\frac{\sin(2k-1)x}{2k-1}$$

so, in particular,

$$s_{2m}(f)\big(\tfrac{\pi}{2m}\big)=\tfrac{4}{\pi}\sum_{k=1}^m\frac{\sin\frac{(2k-1)\pi}{2m}}{(2k-1)}=\tfrac{2}{\pi}R_m$$

Thus

$$\lim_{m\to\infty}s_{2m}(f)\big(\tfrac{\pi}{2m}\big)=\frac{2}{\pi}\lim_{m\to\infty}R_m=\frac{2}{\pi}\int_0^\pi\frac{\sin x}{x}\,dx.$$

(c) Using a computer to approximate the value of $\dfrac{2}{\pi}\displaystyle\int_0^\pi\frac{\sin x}{x}\,dx$, show that there exists $\ell\in\mathbf{Z}^+$ such that for all $m\ge\ell$ we have $\|s_m(f)-f\|_\infty>0.17$.

Solution: Using uniform convergence of power series (allowing term-by-term integration) and the Alternating Series Test, and then using a calculator, we have

$$\begin{aligned}\frac{2}{\pi}\int_0^\pi\frac{\sin x}{x}\,dx&=\tfrac{2}{\pi}\int_0^\pi\Big(1-\tfrac{1}{3!}x^2+\tfrac{1}{5!}x^4-\tfrac{1}{7!}x^7+\cdots\Big)dx=\tfrac{2}{\pi}\Big[x-\tfrac{1}{3\cdot3!}x^3+\tfrac{1}{5\cdot5!}x^5-\tfrac{1}{7\cdot7!}x^7\cdots\Big]_0^\pi\\&=\Big(2-\tfrac{2\pi^3}{2\cdot3!}+\tfrac{2\pi^4}{5\cdot5!}-\tfrac{2\pi^6}{7\cdot7!}+\cdots\Big)>\Big(2-\tfrac{2\pi^2}{3\cdot3!}+\tfrac{2\pi^4}{5\cdot5!}-\tfrac{2\pi^6}{7\cdot7!}\Big)>1.1735737\end{aligned}$$

Choose $\ell\in\mathbf{Z}^+$ so that for $m\ge\ell$ we have $s_{2m}(f)\big(\frac{\pi}{2m}\big)-f\big(\frac{\pi}{2m}\big)>(1.173-1)=0.173$. Then for all $m\ge\ell$ we have $\|s_{2m-1}(f)-f\|_\infty=\|s_{2m}(f)-f\|_\infty\ge\big|s_{2m}(f)\big(\frac{\pi}{2m}\big)-f\big(\frac{\pi}{2m}\big)\big|>0.173$.

(d) (Optional Challenge) Show that $\big\{s_m(f)(x)\big\}$ converges for all x.

Solution: When $x=k\pi$ with $k\in\mathbf{Z}$ we have $s_m(f)(x)=0$ for all x. Suppose that $x\ne k\pi$ for $k\in\mathbf{Z}$. Then

$$\begin{aligned}\sum_{k=0}^n\sin(2k+1)x&=\text{Im}\Big(\sum_{k=0}^n e^{i(2k+1)x}\Big)=\text{Im}\left(\frac{e^{ix}\big(e^{i(n+1)2x}-1\big)}{e^{i\,2x}-1}\right)\\&=\text{Im}\left(\frac{e^{ix}\cdot 2i\,e^{i(n+1)x}\sin(n+1)x}{2i\,e^{ix}\sin x}\right)=\frac{\sin^2(n+1)x}{\sin x}\le\frac{1}{\sin x}.\end{aligned}$$

Since the partial sums $\sum\limits_{k=0}^n\sin(2k+1)x$ are bounded by $\frac{1}{\sin x}$ and the sequence $\big\{\frac{4}{\pi(2k+1)}\big\}$ is decreasing with limit 0, it follows from Dirichlet's Test for Convergence (which most students will not have seen before, so they will need to look it up) that the series $\sum\limits_{k=0}^\infty\frac{4}{\pi(2k+1)}\sin(2k+1)x$ converges.