AMATH/PMATH 331 Real Analysis, Solutions to the Problems for Chapter 7

1: (a) Find an example of a function $f:\mathbf{R}\to\mathbf{R}$ such that $\left|f(y)-f(x)\right|<|y-x|$ for all $x,y\in\mathbf{R}$ with $x\neq y$, but f has no fixed point in $\mathbf{R}$.

Solution: Define $f:\mathbf{R}\to\mathbf{R}$ by $f(x)=x+g(x)$ where $g:\mathbf{R}\to\mathbf{R}$ is any differential function with $g(x)>0$ and $-1<g'(x)<0$ for all $x\in\mathbf{R}$ $\big($for example, $g(x)=\frac{1}{4}\big(\sqrt{x^2+1}-x\big)$ or $g(x)=\frac{1}{2}-\frac{1}{\pi}\tan^{-1}x\big)$. Given $x<y$, by the Mean Value Theorem we can choose c with $x\leq c\leq y$ such that $g(y)-g(x)=g'(c)(y-x)$ and then

$$\left|f(y)-f(x)\right|=\left|(y-x)+(g(y)-g(x))\right|=\left|(y-x)+g'(c)(y-x)\right|=(1+g'(c))(y-x)<(y-x)$$

since $-1<g'(c)<0$. But f has no fixed points because for all $x\in\mathbf{R}$ we have $f(x)=x+g(x)>x$, since $g(x)>0$.

(b) Define $F:\mathcal{C}[0,1]\to\mathcal{C}[0,1]$ by $F(f)(x)=\displaystyle\int_0^x f(t)\,dt$. Show that F is not a contraction map but that $F^2=F\circ F$ is.

Solution: Note that F and F^2 are linear maps on the normed linear space $\mathcal{C}[0,1]$. When f is the constant function $f(x)=1$ we have $F(f)(x)=x$ so that $\|F(f)\|_\infty=1=\|f\|_\infty$, and so F is not a contraction. When $f\in\mathcal{C}[0,1]$ and $F(f)=g$ we have

$$\begin{aligned}\left|g(x)\right|&=\left|\int_0^x f(t)\,dt\right|\leq\int_0^x\left|f(t)\right|dt\leq\int_0^x\|f\|_\infty dt=\|f\|_\infty x\\ \left|F(g)(x)\right|&=\left|\int_0^x g(t)\,dt\right|\leq\int_0^x\left|g(t)\right|dt\leq\int_0^x\|f\|_\infty\,t\,dt=\tfrac{1}{2}\|f\|_\infty x^2\end{aligned}$$

so that $\|F^2(f)\|_\infty=\|F(g)\|_\infty\leq\frac{1}{2}\|f\|_\infty$, and so F^2 is a contraction map with contraction constant $c=\frac{1}{2}$.

(c) Use the Banach Fixed Point Theorem to show that there exists a unique function $f\in\mathcal{C}[0,1]$ such that $f(x)=x+\displaystyle\int_0^x t\,f(t)\,dt$ for all $x\in[0,1]$.

Solution: Define $F:\mathcal{C}[0,1]\to\mathcal{C}[0,1]$ by $F(f)(x)=x+\displaystyle\int_0^x t\,f(t)\,dt$. Note that F is a contraction map because for $f,g\in\mathcal{C}[0,1]$ we have

$$\begin{aligned}\left|F(f)(x)-F(g)(x)\right|&=\left|\left(x+\int_0^x t\,f(t)\,dt\right)-\left(x+\int_0^x t\,g(t)\,dt\right)\right|=\left|\int_0^x t\big(f(t)-g(t)\big)\,dt\right|\\ &\leq\int_0^x\|f-g\|_\infty t\,dt=\tfrac{1}{2}\,\|f-g\|_\infty\,x^2\end{aligned}$$

so that $\|F(f)-F(g)\|_\infty\leq\frac{1}{2}\|f-g\|_\infty$. By the Banach Fixed-Point Theorem, F has a unique fixed point $f\in\mathcal{C}[0,1]$, so there is a unique function $f\in\mathcal{C}[0,1]$ such that $f(x)=x+\displaystyle\int_0^x t\,f(t)\,dt$ for all $x\in[0,1]$.

2: (a) Let $A = \left\{ \sum_{k=1}^{n} f_k(x) g_k(y) \,\middle|\, n \in \mathbf{Z}^+, f_k, g_k \in \mathcal{C}[0,1] \right\}$. Show that A is dense in $\big(\mathcal{C}([0,1]\times[0,1]), d_\infty\big)$.

Solution: It is easy to see that A is a subalgebra of $\mathcal{C}\big([0,1]\times[0,1]\big)$, and A vanishes nowhere because $1 \in A$, and A separates points because $x \in A$ and $y \in A$ (and for $x_1, x_2, y_1, y_2 \in [0,1]$, if $(x_1, y_1) \neq (x_2, y_2)$ then either $x_1 \neq x_2$ or $y_1 \neq y_2$). Thus A is dense in $\mathcal{C}\big([0,1]\times[0,1]\big)$ by the Stone-Weierstrass Theorem.

(b) Let $A = \left\{ \sum_{k=0}^{n} \big(a_k \sin(kx) + b_k \cos(kx)\big) \,\middle|\, 0 \leq n \in \mathbf{Z}, a_k, b_k \in \mathbf{R} \right\}$. Show that A is dense in $\big(\mathcal{C}[0,r], d_\infty\big)$ for every $0 < r < 2\pi$, but A is not dense in $\big(\mathcal{C}[0,2\pi], d_\infty\big)$.

Solution: Let $0 < r < 2\pi$. Note that A is a subalgebra of $\mathcal{C}[0,r]$ because

$$\begin{aligned} \sin(kx)\sin(\ell x) &= \tfrac{1}{2}\Big(\cos\big((k-\ell)x\big) - \cos\big((k+\ell)x\big) \Big), \\ \sin(kx)\cos(\ell x) &= \tfrac{1}{2}\Big(\sin\big((k+\ell)x\big) + \sin\big((k-\ell)x\big) \Big), \\ \cos(kx)\sin(\ell x) &= \tfrac{1}{2}\Big(\sin\big((k+\ell)x\big) - \sin\big((k-\ell)x\big) \Big) \text{ and} \\ \cos(kx)\cos(\ell x) &= \tfrac{1}{2}\Big(\cos\big((k+\ell)x\big) + \cos\big((k-\ell)x\big) \Big), \end{aligned}$$

and A vanishes nowhere because $1 \in A$, and A separates points because $\cos x \in A$ and $\sin x \in A$ and when $x, y \in [0,r]$ with $x \neq y$, either $\cos x \neq \sin x$ or $\cos y \neq \sin y$. Thus A is dense in $\big(\mathcal{C}([0,r]), d_\infty\big)$ by the Stone-Weierstrass Theorem.

The reason that A is not dense in $\mathcal{C}[0,2\pi]$ is that for every $f \in A$ we have $f(0) = f(2\pi)$. When $g \in \mathcal{C}[0,2\pi]$ with $g(0) \neq g(2\pi)$, for every $f \in A$ we have

$$\big|g(0) - g(2\pi)\big| \leq \big|g(0) - f(0) + f(2\pi) - g(2\pi)\big| \leq \big|g(0) - f(0)\big| + \big|f(2\pi) - g(2\pi)\big| \leq 2\|f - g\|_\infty$$

so that $\|f - g\|_\infty \geq \frac{1}{2}\big|g(0) - g(2\pi)\big|$.

(c) Show that there does exist $0 \neq f \in \mathcal{C}[-1,2]$ such that $\displaystyle\int_{-1}^{2} x^{2n} f(x)\,dx = 0$ for all $0 \leq n \in \mathbf{Z}$ but there does not exist $0 \neq f \in \mathcal{C}[-1,2]$ such that $\displaystyle\int_{-1}^{2} x^{3n} f(x)\,dx = 0$ for all $0 \leq n \in \mathbf{Z}$.

Solution: If f is any continuous function whose restriction to $[-1,1]$ is odd and whose restriction to $[1,2]$ is zero (such as the function given by $f(x) = \sin(\pi x)$ for $-1 \leq x \leq 1$ and $f(x) = 0$ for $1 \leq x \leq 2$) then we have $\displaystyle\int_{-1}^{2} x^{2n} f(x)\,dx = 0$ for all $0 \leq n \in \mathbf{Z}$.

Let $A = \left\{ \sum_{k=0}^{n} c_k x^{3k} \,\middle|\, 0 \leq n \in \mathbf{Z}, c_k \in \mathbf{R} \right\}$. Note that A is a subalgebra of $\mathcal{C}[-1,2]$ and A vanishes nowhere because $1 \in A$, and A separates points because $x^3 \in A$ and x^3 is strictly increasing on $[-1,2]$, and so A is dense in $\mathcal{C}[-1,2]$ by the Stone-Weierstrass Theorem. Let $f \in \mathcal{C}[-1,2]$ with $\displaystyle\int_{-1}^{2} x^{3n} f(x)\,dx = 0$ for all $0 \leq n \in \mathbf{Z}$ and note that $\int_{-1}^{2} pf = 0$ for every $p \in A$. Since A is dense in $\mathcal{C}[-1,2]$ we can choose a sequence $(p_n)_{n\geq 1}$ in A with $p_n \to f$ in $\mathcal{C}[-1,2]$. Then $p_n \to f$ uniformly on $[-1,2]$, so $p_n f \to f^2$ uniformly on $[-1,2]$, and hence $\int_{-1}^{2} f^2 = \lim_{n\to\infty} \int_{-1}^{2} p_n f = \lim_{n\to\infty} 0 = 0$. Since f is continuous on $[-1,2]$ and $\int_{-1}^{2} f^2 = 0$, it follows that $f = 0$.