AMATH/PMATH 331 Real Analysis, Solutions to the Problems for Chapter 6

1: (a) Let $A = \left\{ x \in \mathbf{R}^2 \,\middle|\, \|x\| = \frac{n+1}{n} \text{ for some } n \in \mathbf{Z}^+ \right\}$. Determine whether A is complete and whether A is compact.

Solution: Note that A is not closed in $\mathbf{R}^2$ because for the sequence $(x_n)_{n\geq 1}$ in $\mathbf{R}^2$ given by $x_n = \left(\frac{n+1}{n}, 0\right)$, we have $\|x_n\| = \frac{n+1}{n}$ so that each $x_n \in A$, and we have $\lim\limits_{n\to\infty} x_n = e_1 = (1,0)$ in $\mathbf{R}^2$, but $e_1 \notin A$ $\big($since for all $n \in \mathbf{Z}^+$ we have $\frac{n+1}{n} > 1 = \|e_1\|\big)$. Since $\mathbf{R}^2$ is complete and A is not closed in $\mathbf{R}^2$, it follows from Theorem 6.4 that A is not complete. Since A is not closed in $\mathbf{R}^2$, it follows from Theorem 6.21 that A is not compact.

(b) Let A be the set of points $(a,b,c,d) \in \mathbf{R}^4$ such that the points $(0,0)$, (a,b) and (c,d) are the vertices of a right-angled triangle in $\mathbf{R}^2$ whose area is equal to 1. Determine whether A is complete and whether A is compact.

Solution: Consider the triangle T in $\mathbf{R}^2$ with vertices at $(0,0)$, (a,b) and (c,d). Triangle T has a right angle at $(0,0)$, with area equal to 1, if and only if $(a,b)\bullet(c,d) = 0$, that is $ac + bd = 0$, and $\frac{1}{2}\big|(a,b)\big|\,\big|(c,d)\big| = 1$, that is $(a^2+b^2)(c^2+d^2) = 4$. Triangle T has a right angle at (a,b), with area equal to 1, if and only if $(a,b)\bullet\big((c,d)-(a,b)\big) = 0$, that is $a(c-a)+b(d-b) = 0$, and $\frac{1}{2}\big|(a,b)\big|\,\big|(c,d)-(a,b)\big| = 1$, that is $(a^2+b^2)\big((c-a)^2+(d-b)^2\big) = 4$. Similarly, T has a right angle at (c,d) with area equal to 1 if and only if $c(a-c)+d(b-d) = 0$ and $(c^2+d^2)\big((a-c)^2+(b-d)^2\big) = 4$. Define $f,g,h,k,l,m : \mathbf{R}^4 \to \mathbf{R}$ by

$$\begin{aligned}
f(a,b,c,d) &= ac + bd\,,\\
g(a,b,c,d) &= (a^2+b^2)(c^2+d^2)\,,\\
h(a,b,c,d) &= a(c-a)+b(d-b)\,,\\
k(a,b,c,d) &= (a^2+b^2)\big((c-a)^2+(d-b)^2\big)\,,\\
\ell(a,b,c,d) &= c(a-c)+d(b-d)\,,\\
m(a,b,c,d) &= (c^2+d^2)\big((a-c)^2+(b-d)^2\big)\,.
\end{aligned}$$

These functions are all continuous (they are polynomials). For $x = (a,b,c,d) \in \mathbf{R}^6$, we have

$$x \in A \iff \big(f(x) = 0 \text{ and } g(x) = 4\big) \text{ or } \big(h(x) = 0 \text{ and } k(x) = 4\big) \text{ or } \big(\ell(x) = 0 \text{ and } m(x) = 4\big)$$

and hence

$$A = B \cup C \cup D$$

where

$$\begin{aligned}
B &= \left\{x \in \mathbf{R}^4 \,\middle|\, f(x) = 0 \text{ and } g(x) = 4\right\} = f^{-1}(0) \cap g^{-1}(4)\,,\\
C &= \left\{x \in \mathbf{R}^4 \,\middle|\, h(x) = 0 \text{ and } k(x) = 4\right\} = h^{-1}(0) \cap k^{-1}(4)\,,\\
D &= \left\{x \in \mathbf{R}^4 \,\middle|\, \ell(x) = 0 \text{ and } m(x) = 4\right\} = \ell^{-1}(0) \cap m^{-1}(4)\,.
\end{aligned}$$

Since f and g are continuous, and $\{0\}$ and $\{4\}$ are closed in $\mathbf{R}$, it follows from Part 2 of Theorem 5.29 (the topological characterization of continuity) that $f^{-1}(0)$ and $g^{-1}(4)$ are closed in $\mathbf{R}^4$, and hence the set $B = f^{-1}(0) \cap g^{-1}(4)$ is closed in $\mathbf{R}^4$ by Part 2 of Theorem 4.36 (Basic Properties of Closed Sets). Similarly, the sets C and D are both closed, and hence the set $A = B \cup C \cup D$ is closed by Part 3 of Theorem 4.36. On the other hand, the set A is not bounded because for $r > 0$ and $x = \left(r, 0, 0, \frac{2}{r}\right)$ we have $x \in A$ and $\|x\| = \sqrt{r^2 + \frac{4}{r^2}} > r$. Since $\mathbf{R}^4$ is complete and A is closed in $\mathbf{R}^4$, it follows (from Theorem 6.4) that A is complete, and since A is not bounded, it follows (from Theorem 6.21) that A is not compact.

2: (a) Let $M_2(\mathbf{R})$ be the set of 2×2 matrices with entries in $\mathbf{R}$ and let $O_2(\mathbf{R}) = \{A\in M_2(\mathbf{R})\,\big|\,A^TA=I\}$. Define $F:\mathbf{R}^4\to M_2(\mathbf{R})$ by $F(x,y,z,w)=\begin{pmatrix} x & z\\ y & w\end{pmatrix}$ and let $A=\Big\{(x,y,z,w)\in\mathbf{R}^4\,\Big|\,F(A)\in O_2(\mathbf{R})\Big\}$. Show that A is compact.

Solution: For $(x,y,z,w)\in\mathbf{R}^4$ and $A=F(A)$ we have

$$(x,y,z,w)\in A \iff F(A)^TF(A)=I \iff \begin{pmatrix} x^2{+}y^2 & xz{+}yw\\ yw{+}yw & z^2{+}w^2\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$
$$\iff \Big(x^2+y^2=1 \text{ and } xz+yw=0 \text{ and } z^2+w^2=1\Big).$$

Define $f,g,h:\mathbf{R}^4\to\mathbf{R}$ by $f(x,y,z,w)=x^2+y^2$, $g(x,y,z,w)=xz+yw$ and $h(x,y,z,w)=z^2+w^2$. Note that f, g and h are continuous (they are polynomials) and we have

$$A=f^{-1}(1)\cap g^{-1}(0)\cap h^{-1}(1).$$

Since $\{0\}$ and $\{1\}$ are closed in $\mathbf{R}$ and f, g and h are continuous, it follows (from Part 2 of Theorem 5.29) that each of the sets $f^{-1}(1)$ and $g^{-1}(0)$ and $h^{-1}(1)$ are closed, and hence it follows (from Theorem 4.36) that $A=f^{-1}(1)\cap g^{-1}(0)\cap h^{-1}(1)$ is closed. Also note that A is bounded because when $u=(x,y,z,w)\in A$ we have $x^2+y^2=1$ and $z^2+w^2=1$ so that $\|u\|=\sqrt{x^2+y^2+z^2+w^2}=\sqrt{2}$. Since A is closed and bounded in $\mathbf{R}^4$, it is compact by the Heine Borel Theorem.

(b) Recall from linear algebra (or verify) that the space $M_{n\times m}(\mathbf{R})$ of $n\times m$ matrices with entries in $\mathbf{R}$ is an inner-product space with inner product given by $\langle A,B\rangle=\text{trace}(B^TA)=\sum\limits_{k=1}^{n}\sum\limits_{\ell=1}^{m}A_{k,\ell}B_{k,\ell}$, and with standard orthonormal basis $\{E_{k,\ell}\,\big|\,1\le k\le n, 1\le\ell\le m\}$ where $E_{k,\ell}$ is the $n\times m$ matrix whose (k,ℓ) entry is equal to 1 and all other entries are zero, and the linear map $L=L_{n,m}:M_{n\times m}(\mathbf{R})\to\mathbf{R}^{nm}$ given by $L(E_{k,\ell})=e_{(k-1)n+\ell}$ is an inner product space isomorphism. Show that the set $S=\{A\in M_{n\times m}(\mathbf{R})\,\big|\,A^TA=I\}$ is compact.

Solution: For $X\in M_{n\times m}(\mathbf{R})$, we have $(X^TX)_{k,\ell}=\sum\limits_{j=1}^{n}(X^T)_{k,j}X_{j,\ell}=\sum\limits_{j=1}^{n}X_{j,k}X_{j,\ell}$. Let $X\in S$. Then $X^TX=I$ so we have $(X^TX)_{\ell,\ell}=1$ for all indices ℓ, that is $\sum\limits_{k=1}^{n}(X_{k,\ell})^2=1$ for all $1\le\ell\le m$. Thus we have $\|X\|^2=\sum\limits_{k,\ell=1}^{n}(X_{k,\ell})^2=\sum\limits_{\ell=1}^{n}1=n$ so that $\|X\|=\sqrt{n}$. Since $\|X\|=\sqrt{n}$ for all $X\in S$, S is bounded.

We claim that S is closed. Define $F:M_{n\times m}\to M_{m\times m}(\mathbf{R})$ by $F(X)=X^TX$ so that $S=F^{-1}(I)$, and let G be the composite $G=L_{m,m}F\,L_{n,m}^{-1}:\mathbf{R}^{nm}\to\mathbf{R}^m$. The map G is continuous if and only if each of its component maps $G_j:\mathbf{R}^{nm}\to\mathbf{R}$ is continuous. For $j=(k-1)m+\ell$ with $1\le k,\ell\le m$, and for $x=(x_1,\cdots,x_{n,m})^T\in\mathbf{R}^{n,m}$ and $X=L_{n,m}^{-1}(x)$ so that $X_{j,k}=x_{(j-1)n+k}$, we have

$$G_j(x)=\big(G(x)\big)_{(k-1)m+\ell}=\big(L_{m,m}F\,L_{n,m}^{-1}(x)\big)_{(k-1)m+\ell}=\big(L_{m,m}F(X)\big)_{(k-1)m+\ell}=F(X)_{k,\ell}$$
$$=(X^TX)_{k,\ell}=\sum_{j=1}^{n}X_{j,k}X_{j,\ell}=\sum_{j=1}^{n}x_{(j-1)n+k}\,x_{(j-1)n+\ell}$$

so each component function G_j is continuous (it is a polynomial of degree 2) and hence G is continuous. Since G is continuous, so is $F=L_{m,m}^{-1}G\,L_{n,m}$. Since $F:M_{n\times m}(\mathbf{R})\to M_{m\times m}(\mathbf{R})$ is continuous and $\{I\}$ is closed in $M_{m\times m}(\mathbf{R})$ (indeed if X is any metric space and $a\in X$ the $\{a\}$ is closed because if $b\in\{a\}^c$ so $b\ne a$ and $r=d(a,b)>0$, then we have $a\notin B(b,r)$ so $B(b,r)\subseteq\{a\}^c$ hence $\{a\}^c$ is open), it follows that $S=F^{-1}(I)$ is closed in $M_{n\times m}(\mathbf{R})$, as claimed.

Finally, note that since S is closed and bounded in $M_{n\times m}(\mathbf{R})$, it follows (from Theorem 6.35) that S is compact.

3: (a) Show that (ℓ_∞, d_∞) is complete.

Solution: Let $(a_n)_{n\geq 1}$ be a Cauchy sequence in ℓ_∞. For each $n \in \mathbf{Z}^+$, a_n is a bounded sequence of real numbers, say $a_n = (a_{n,k})_{k\geq 1}$. Fix an index $k \in \mathbf{Z}^+$ and let $\epsilon > 0$. Since $(a_n)_{n\geq 1}$ is Cauchy in ℓ_∞, we can choose $N \in \mathbf{Z}^+$ such that $n, m \geq N \Longrightarrow \|a_n - a_m\|_\infty < \epsilon$. Then for $n, m \geq N$ we have $|a_{n,j} - b_{n,j}| < \epsilon$ for all $j \in \mathbf{Z}^+$ so, in particular, $|a_{n,k} - b_{n,k}| < \epsilon$. This shows that for each $k \in \mathbf{Z}^+$, the sequence $(x_{n,k})_{k\geq 1}$ is a Cauchy sequence in $\mathbf{R}$, so it converges. For each $k \in \mathbf{Z}^+$, let $b_k = \lim\limits_{n\to\infty} b_{n,k} \in \mathbf{R}$, and then let $b = (b_k)_{k\geq 1}$.

We claim that $b \in \ell_\infty$ (that is, the sequence $b = (b_k)_{k\geq 1}$ is bounded in $\mathbf{R}$). Since $(a_n)_{n\geq 1}$ is Cauchy in ℓ_∞, it is bounded in ℓ_∞, so we can choose $M \geq 0$ such that $\|a_n\|_\infty \leq M$ for all indices $n \in \mathbf{Z}^+$. Then for all $k, n \in \mathbf{Z}^+$ we have $|a_{n,k}| \leq \|a_n\|_\infty \leq M$ and hence, for all $k \in \mathbf{Z}^+$, $|b_k| = \big|\lim\limits_{n\to\infty} b_{n,k}\big| = \lim\limits_{n\to\infty} |b_{n,k}| \leq M$. Thus the sequence $(b_k)_{k\geq 1}$ is bounded in $\mathbf{R}$, that is $b \in \ell_\infty$, as claimed.

Finally, we claim that $a_n \to b$ in ℓ_∞. Let $\epsilon > 0$. Choose $N \in \mathbf{Z}^+$ so that $n, m \geq N \Longrightarrow \|a_n - a_m\|_\infty < \epsilon$. Then for $n, m \geq N$ we have $|a_{n,k} - a_{m,k}| < \epsilon$ for all indices $k \in \mathbf{Z}^+$. It follows that for all $n \geq N$ and for all $k \in \mathbf{Z}^+$ we have $|a_{n,k} - b_k| = \lim\limits_{m\to\infty} \big|a_{n,k} - a_{m,k}\big| \leq \epsilon$ and hence, for all $n \geq N$, we have $\|a_n - b\|_\infty \leq \epsilon$. This shows that $a_n \to b$ in ℓ_∞, as claimed.

(b) Show that $\big(\ell_2, d_2\big)$ is complete.

Solution: Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in $\big(\ell_2, d_2\big)$, say $x_n = (x_{n,k})_{k\geq 1}$. Note that for each fixed $k \in \mathbf{Z}^+$, the sequence $(x_{n,k})_{n\geq 1}$ is Cauchy; indeed given $\epsilon > 0$ we can choose $N \in \mathbf{Z}^+$ so that for all $n, m \in \mathbf{Z}^+$ we have $n, m \geq N \Longrightarrow \|x_n - x_m\|_2 < \epsilon$, and then for $n, m \geq N$ we have

$$|x_{n,k} - x_{m,k}| \leq \Big(\sum_{i=1}^{\infty} (x_{n,i} - x_{m,i})^2\Big)^{1/2} = \|x_n - x_m\|_2 < \epsilon\,.$$

Since $(x_{n,k})_{n\geq 1}$ is a Cauchy sequence in $\mathbf{R}$, and since $\mathbf{R}$ is complete, this sequence converges. Let

$$a = (a_k)_{k\geq 1}, \text{ where } a_k = \lim_{n\to\infty} x_{n,k}\,.$$

We claim that $a \in \ell_2$, that is $\sum\limits_{k=1}^{\infty} |a_k|^2 < \infty$. For $K \in \mathbf{Z}^+$ we have

$$\sum_{k=1}^{K} |a_k|^2 = \sum_{k=1}^{K} \big|\lim_{n\to\infty} x_{n,k}\big|^2 = \sum_{k=1}^{K} \lim_{n\to\infty} {x_{n,k}}^2 = \lim_{n\to\infty} \sum_{k=1}^{K} {x_{n,k}}^2 \leq \lim_{n\to\infty} \sum_{k=1}^{\infty} {x_{n,k}}^2 = \lim_{n\to\infty} {\|x_n\|_2}^2\,,$$

so it suffices to show that the sequence $\big(\|x_n\|_2\big)$ converges in $\mathbf{R}$. And since $\big|\|x_n\|_2 - \|x_m\|_2\big| \leq \|x_n - x_m\|_2$ (by the Triangle Inequality) we see that $\big(\|x_n\|_2\big)$ is Cauchy in $\mathbf{R}$, so it does converge.

Finally, we claim that $x_n \to a$ in $\big(\ell_2, d_2\big)$. Let $\epsilon > 0$. Choose $N \in \mathbf{Z}^+$ so that for all $n, m \in \mathbf{Z}^+$ we have

$$n, m \geq N \Longrightarrow \big\|x_n - x_m\big\|_2 < \tfrac{\epsilon}{2}\ \text{, that is } \sum_{k=1}^{\infty} (x_{n,k} - x_{m,k})^2 < \tfrac{\epsilon^2}{4}\,.$$

Let $n \in \mathbf{Z}^+$. Then for all $K \in \mathbf{Z}^+$ we have

$$\sum_{k=1}^{K} (x_{n,k} - a_k)^2 = \sum_{k=1}^{K} \Big(x_{n,k} - \lim_{m\to\infty} x_{m,k}\Big)^2 = \lim_{m\to\infty} \sum_{k=1}^{K} (x_{n,k} - x_{m,k})^2 \leq \lim_{m\to\infty} \sum_{k=1}^{\infty} (x_{n,k} - x_{m,k})^2 \leq \tfrac{\epsilon^2}{4}$$

and so

$$\|x_n - a\|_2 = \Big(\sum_{k=1}^{\infty} (x_{n,k} - x_{m,k})^2\Big)^{1/2} \leq \tfrac{\epsilon}{2} < \epsilon\,.$$

4: For each of the following sets A, determine whether A is complete and whether A is compact.

(a) $A = \overline{B}_2(0,1) = \left\{a = (a_n)_{n\geq 1} \in \ell_2 \,\middle|\, ||a||_2 \leq 1\right\} \subseteq \ell_2$, using the metric d_2.

Solution: Let $E = \{e_1, e_2, e_3, \cdots\}$, let $U_0 = \ell_2 \setminus E$, let $U_n = B(e_n, 1) = \left\{x \in \ell_2 \middle| ||x - e_n||_2 < 1\right\}$ for $n \in \mathbf{Z}^+$. and let $\mathcal{U} = \left\{U_0, U_1, U_2, \cdots\right\}$. Note that E is closed (because for all $k \neq \ell$ we have $||e_k - e_\ell||_2 = \sqrt{2}$, so every Cauchy sequence in E is eventually constant) and so U_0 is open, and so $\mathcal{U}$ is an open cover of B. But $\mathcal{U}$ has no finite subcover, indeed $\mathcal{U}$ has no proper subcover, because the point $0 \in B$ only lies in the set U_0 and for each $k \in \mathbf{Z}^+$, the point $e_k \in B$ only lies in the set U_k (when $n \in \mathbf{Z}^+$ with $n \neq k$ we have $||e_k - e_n||_2 = \sqrt{2}$ so $e_k \notin B(e_n, 1) = U_n$).

(b) $A = \left\{a = (a_n)_{n\geq 1} \in \ell_1 \,\middle|\, |a_n| \leq \frac{1}{n} \text{ for all } n \in \mathbf{Z}^+\right\} \subseteq \ell_1$, using the metric d_1.

Solution: We claim that A is closed in ℓ_1. Let $a = (a_n)_{n\geq 1} \in A^c = \ell_1 \setminus A$. Choose $m \in \mathbf{Z}^+$ so that $|a_m| > \frac{1}{m}$ and let $r = |a_m| - \frac{1}{m}$ and note that $r > 0$. We claim that $B(a, r) \subseteq A^c$. Let $x = (x_n)_{n\geq 1} \in B(a, r)$. Since $|a_m| = \big|a_m - x_m + x_m\big| \leq |a_m - x_m| + |x_m|$, we have

$$|a_m| - |x_m| \leq |a_m - x_m| \leq \sum_{k=0}^{\infty} |a_k - x_k| = \|a - x\|_1 < r = |a_m| - \tfrac{1}{m}$$

and hence $|x_m| > \frac{1}{m}$ so that $x \in A^c$. Thus $B(a, r) \subseteq A^c$, showing that that A^c is open, hence A is closed, as claimed. Since ℓ_1 is complete and A is closed in ℓ_1, it follows that A is complete. On the other hand, A is not bounded because given $r > 0$, since $\sum\limits_{k=1}^{\infty} \frac{1}{k} = \infty$ we can choose $n \in \mathbf{Z}^+$ such that $\sum\limits_{k=1}^{n} \frac{1}{k} > r$, and then we can let $x \in \ell_1$ be given by $x = \sum\limits_{k=1}^{n} \frac{1}{k} e_k = \left(\frac{1}{1}, \frac{1}{2}, \cdots, \frac{1}{n}, 0, 0, \cdots\right)$, and then we have $x \in A$ but $\|x\|_1 = \sum\limits_{k=1}^{n} \frac{1}{k} > r$. Since A is not bounded, it is not compact.

(c) $A = \left\{a = (a_n)_{n\geq 1} \in \ell_2 \,\middle|\, |a_n| \leq \frac{1}{n+1} \text{ for all } n \in \mathbf{Z}^+\right\} \subseteq \ell_2$, using the metric d_2.

Solution: We claim that A is closed in (ℓ_2, d_2), and hence A is complete since (ℓ_2, d_2) is complete. Let $a = \langle a_n \rangle \in \ell_2 \setminus A$. Choose $N \in \mathbf{N}$ so that $|a_N| > \frac{1}{N+1}$. Let $r = |a_N| - \frac{1}{N+1}$. We claim that $B(a, r) \cap A = \emptyset$. Let $b = \langle b_n \rangle \in B(a, r)$. Then we have

$$|a_N| - |b_N| \leq |b_N - a_N| \leq \sqrt{\sum_{n=0}^{\infty} (b_n - a_n)^2} = \big|b - a\big|_2 < r = |a_N| - \tfrac{1}{N+1}$$

so $|b_N| > \frac{1}{N+1}$, and hence $b \notin B(a, r)$. Thus A is closed in (ℓ_2, d_2), hence complete.

We claim that A is totally bounded. Let $\epsilon > 0$. Choose $N \in \mathbf{N}$ so that $\sum\limits_{n=N}^{\infty} \left(\frac{1}{N+1}\right)^2 < \frac{\epsilon^2}{2}$, and then let $\delta = \sqrt{\frac{\epsilon^2}{2N}}$. For each $n = 0, 1, \cdots, N-1$ choose points $t_{n,1}, t_{n,2}, \cdots, t_{n,m_n} \in \left[-\frac{1}{n+1}, \frac{1}{n+1}\right]$ such that $\left[-\frac{1}{n+1}, \frac{1}{n+1}\right] \subset \bigcup\limits_{i=1}^{m_n} B(t_{n,i}, \delta)$, then let $A_n = \{t_{n,1}, t_{n,2}, \cdots, t_{n,m_n}\}$. Let $P = A_0 \times A_1 \times \cdots \times A_{N-1}$. For each $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{N-1}) \in P$, Let $a_\alpha = \langle a_{\alpha,n} | n \in \mathbf{N}\rangle$ be the sequence $a_\alpha = \langle \alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_{N-1}, 0, 0, 0, \cdots\rangle$. We claim that $A \subset \bigcup\limits_{\alpha\in P} B(a_\alpha, \epsilon)$, and hence A is totally bounded. Let $b = \langle b_n \rangle \in A$. For each $n < N$ choose $\alpha_n \in A_n$ so that $b_n \in B(\alpha_n, \delta)$, then let $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{N-1}) \in P$. Then we have

$$\begin{aligned}\big|b - a_\alpha\big|_2 &= \sqrt{\sum_{n=0}^{\infty} (b_n - a_{\alpha,n})^2} = \sqrt{\sum_{n=0}^{N-1} (b_n - \alpha_n)^2 + \sum_{n=N}^{\infty} {b_n}^2} \\ &\leq \sum_{n=0}^{N-1} \delta^2 + \sum_{n=N}^{\infty} \left(\frac{1}{N+1}\right)^2 < \sqrt{N\delta^2 + \frac{\epsilon^2}{2}} = \epsilon\,.\end{aligned}$$

Thus $b \in B(\alpha_n, \epsilon)$, so A is totally bounded. Since A is complete and totally bounded, A is compact.

5: (a) Show that the closed unit ball $\overline{B}_\infty(0,1)$ is not compact in $\mathcal{C}[0,1]$, using the metric d_∞.

Solution: Let $p(x) = \begin{cases} 1-x^2 & , \text{ for } |x| \le 1 \\ 0 & , \text{ for } |x| \ge 1 \,. \end{cases}$ For each $n \in \mathbf{N}$ let $f_n(x) = p\big(2^{n+2}(x - \frac{1}{2^n})\big)$, so that $f_n(x)$ is a continuous bump function of height 1 centred at $\frac{1}{2^n}$ of width $\frac{1}{2^{n+1}}$ (so the bumps of f_n and f_m do not overlap when $n \ne m$). We have $\|f_n\|_\infty = f_n\big(\frac{1}{2^n}\big) = 1$ so that $f_n \in \overline{B}(0,1)$. Notice that for $n \ne m$ we have $\|f_n - f_m\|_\infty = 1$ $\big($since $f_n\big(\frac{1}{2^n}\big) = 1$ and $f_m\big(\frac{1}{2^n}\big) = 0$ and $f_n(x), f_m(x) \in [0,1]$ for all $x\big)$, so no subsequence of (f_n) converges uniformly on $[0,1]$, that is no subsequence of (f_n) converges in the metric space $\mathcal{C}[0,1]$ using d_∞. Thus $\mathcal{C}[0,1]$ is not compact by Part 3 of Theorem 6.38.

(b) Show that $\mathcal{C}[-1,1]$ is not complete using the metric d_1.

Solution: For each $n \in \mathbf{Z}^+$, define $f_n : [-1,1] \to \mathbf{R}$ by $f_n(x) = x^{\frac{1}{2n-1}}$. Note that each f_n is continuous on $[-1,1]$, and the sequence $(f_n)_{n\ge1}$ is Cauchy in $\big(\mathcal{C}[-1,1], d_1\big)$ because for $m \ge n \ge N$ we have

$$\begin{aligned} \|f_n - f_m\|_1 &= \int_{x=-1}^1 \big|f_n(x) - f_m(x)\big|\,dx = 2\int_{x=0}^1 x^{\frac{1}{2m-1}} - x^{\frac{1}{2n-1}}\,dx \\ &= 2\Big[\tfrac{2m-1}{2m}\,x^{\frac{2m+1}{2m-1}} - \tfrac{2n-1}{2n}\,x^{\frac{2n+1}{2n-1}}\Big]_{x=0}^1 = \tfrac{2m-1}{m} - \tfrac{2n-1}{n} = \tfrac{1}{n} - \tfrac{1}{m} \le \tfrac{1}{N}. \end{aligned}$$

Note that for each $x \in [-1,1]$ we have $\lim\limits_{n\to\infty} f_n(x) = g(x)$ in $\mathbf{R}$ where $g(x) = -1$ for $x < 0$, $g(x) = 1$ for $x > 0$ and $g(0) = 0$ $\big($so we have $f_n \to g$ pointwise on $[-1,1]\big)$. Suppose, for a contradiction, that $(f_n)_{n\ge1}$ converges in $\mathcal{C}[-1,1]$, and let $h = \lim\limits_{n\to\infty} f_n$ in $\mathcal{C}[-1,1]$. Note that the restriction of h to $[0,1]$ is continuous. Let $\epsilon > 0$. Choose $n \in \mathbf{Z}^+$ such that $\|f_n - h\|_1 < \frac{\epsilon}{2}$ and also $\frac{1}{2n} < \frac{\epsilon}{2}$. Then

$$\begin{aligned} \int_{x=0}^1 \big|h(x) - 1\big|\,dx &\le \int_{x=0}^1 \big|h(x) - f_n(x)\big| + \big|f_n(x) - 1\big|\,dx \le \int_{x=-1}^1 \big|h(x) - f_n(x)\big|\,dx + \int_{x=0}^1 \big|f_n(x) - 1\big|\,dx \\ &= \|h - f_n\|_1 + \int_{x=0}^1 1 - x^{\frac{1}{2n-1}}\,dx = \|h - f_n\|_1 + \tfrac{1}{2n} < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since $\int_{x=0}^1 \big|h(x) - 1\big|\,dx < \epsilon$ for every $\epsilon > 0$, it follows that $\int_{x=0}^1 \big|h(x) - 1\big|\,dx = 0$ and, since the function $h(x) - 1$ is continuous on $[0,1]$, it follows that $h(x) - 1 = 0$ for all $x \in [0,1]$. Thus we have $h(x) = 1$ for all $x \in [0,1]$. A similar argument shows that $h(x) = -1$ for all $x \in [-1,0]$. But this is not possible since we cannot have $h(0) = 1$ and $h(0) = -1$.

6: (a) Let X be a metric space, let $A \subseteq X$ be compact, and let S be an open cover for A in X. Show that there exists $r > 0$ with the property that for every $a \in A$ there exists $U \in S$ such that $B(a, r) \subseteq U$.

Solution: For each $a \in A$, since S is an open cover for A we can choose $U_a \in S$ with $a \in U_a$ and then, since U_a is open we can choose $r_a > 0$ so that $B(a, 2r_a) \subseteq U_a$. Note that the set $T = \{B(a, r_a) \big| a \in A\}$ is an open cover for A. Since A is compact, we can choose a finite subcover, say $\{B(a_1, r_{a_1}), \cdots, B(a_n, r_{a_n})\}$ of T for A, with each $a_i \in A$. Let $r = \min\{r_{x_1}, \cdots, r_{x_n}\}$. We claim that for every $a \in A$ there exists $U \in S$ such that $B(a, r) \subseteq U$. Let $a \in A$. Choose an index k such that $a \in B(a_k, r_{a_k})$, and let $U = U_{a_k} \in S$. For all $x \in B(a, r)$ we have $d(x, a_k) \leq d(x, a) + d(a, a_k) \leq r + r_{a_k} \leq 2r_{a_k}$ and hence $x \in B(a_k, 2r_{a_k}) \subseteq U_{a_k} = U$. This shows that $B(a, r) \subseteq U$, as required.

(b) Let X be a compact metric space. Let $(f_n)_{n \geq 1}$ be a sequence in $\mathcal{C}(X)$ which converges pointwise to a function $f \in \mathcal{C}(X)$. Show that if $\big(f_n(x)\big)_{n \geq 1}$ is increasing for every $x \in X$, then the convergence is uniform.

Solution: Let $g_n(x) = f(x) - f_n(x)$. Then $\big(g_n(x)\big)$ is decreasing for all $x \in X$ and $g_n \to 0$ pointwise on X. We need to show that $g_n \to 0$ uniformly. Let $\epsilon > 0$. For each $a \in X$, since $g_n(a) \to 0$ we can choose n_a so that $g_{n_a}(a) < \frac{\epsilon}{2}$, and then since g_{n_a} is continuous we can choose $\delta_a > 0$ so that for all $x \in X$ we have

$$d(x, a) < \delta_a \Longrightarrow \big|g_{n_a}(x) - g_{n_a}(a)\big| < \tfrac{\epsilon}{2}\,.$$

Then for all $x \in X$ with $d(x, a) < \delta_a$ we have $\big|g_{n_a}(x)\big| \leq \big|g_{n_a}(x) - g_{n_a}(a)\big| + \big|g_{n_a}(a)\big| < \epsilon$. Let $x \in X$ and let $n \geq N$. Choose i so that $x \in B\big(a_i, \delta_{a_i}\big)$. Since $\big(g_n(x)\big)$ is decreasing and $n \geq N \geq N_i$, we have $g_n(x) \leq g_{n_i}(x) < \epsilon$. Thus $g_n \to 0$ uniformly on X, as required.

(c) Show that the requirements in Part (b) that X is compact and that (f_n) is increasing are both necessary.

Solution: To see that the requirement that X is compact is necessary, take $X = (0, 1)$ and let $f_n(x) = -x^n$. Then $\big(f_n(x)\big)$ is increasing for all $x \in (0, 1)$ and $f_n \to 0$ pointwise in $(0, 1)$, but the convergence is not uniform.

To see that the requirement that $\big(f_n(x)\big)$ is increasing is necessary, take $X = [0, 1]$ and let f_n be the bump functions used in 5(a). Then $f_n \to 0$ pointwise on $[0, 1]$, but the convergence is not uniform.

7: (Absolute convergence implies convergence) Let X be a normed linear space. For a sequence $(x_k)_{k\geq 1}$ in X, the n^{th} *partial sum* of $(x_k)_{k\geq 1}$ is the element $s_n = \sum_{k=1}^{n} x_k \in X$, the *series* $\sum_{k=1}^{\infty} x_k$ is, by definition, equal to the sequence of partial sums $(s_n)_{n\geq 1}$, we say the series $\sum_{k=1}^{\infty} x_k$ *converges* in X when the sequence of partial sums $(s_n)_{n\geq 1}$ converges in X and then the *sum* of the series $\big($also denoted by $\sum_{k=1}^{\infty} x_k\big)$ is defined to be the limit of the sequence of partial sums in X. Show that X is complete if and only if X has the property that for every sequence $(x_k)_{k\geq 1}$ in X, if $\sum_{k=1}^{\infty} \|x_k\|$ converges in $\mathbf{R}$ then $\sum_{k=1}^{\infty} x_k$ converges in X.

Solution: Suppose that X is complete. Let $(x_k)_{k\geq 1}$ be a sequence in X such that $\sum_{k=1}^{\infty} \|x_k\|$ converges in $\mathbf{R}$. For each $n \in \mathbf{Z}^+$, let $t_n = \sum_{k=1}^{n} \|x_k\| \in \mathbf{R}$ and let $s_n = \sum_{k=1}^{n} x_k \in X$. Let $\epsilon > 0$. Since $\sum_{k=1}^{n} \|x_k\|$ converges in $\mathbf{R}$, the sequence $(t_n)_{n\geq 1}$ is Cauchy in $\mathbf{R}$, so we can choose $N \in \mathbf{Z}^+$ such that for $m > n \geq N$ we have $\sum_{k=n+1}^{m} \|x_k\| = |t_m - t_n| < \epsilon$. Then for $m > n \geq N$ we have $\|s_m - s_n\| = \big\| \sum_{k=n+1}^{m} x_k \big\| \leq \sum_{k=n+1}^{m} \|x_k\| < \epsilon$. This shows that the sequence $(s_n)_{n\geq 1}$ is Cauchy in X, and so it converges in X because X is complete.

Suppose, conversely, that X has the property that for every sequence $(y_k)_{k\geq 1}$ in X, if $\sum_{k=1}^{\infty} \|y_k\|$ converges in $\mathbf{R}$ then $\sum_{k=1}^{\infty} y_k$ converges in X. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in X. Since $(x_n)_{n\geq 1}$ is Cauchy, we can choose $n_1 \in \mathbf{Z}^+$ such that $k, \ell \geq n_1 \implies \|x_k - x_\ell\| < \frac{1}{2}$, then we can choose $n_2 > n_1$ such that $k, \ell \geq n_2 \implies \|x_k - x_\ell\| < \frac{1}{2^2}$, then we can choose $n_3 > n_2$ so that $k, \ell \geq n_3 \implies \|x_k - x_\ell\| < \frac{1}{2^3}$ and so on, to obtain integers n_k with $1 \leq n_1 < n_2 < n_3 < \cdots$ such that $i, j \geq n_k \implies \|x_i - x_j\| < \frac{1}{2^k}$. For each $k \in \mathbf{Z}^+$, let $y_k = x_{n_{k+1}} - x_{n_k}$. Note that

$$\sum_{k=1}^{\infty} \|y_k\| = \sum_{k=1}^{\infty} \|x_{n_{k+1}} - x_{n_k}\| < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

Since $\sum_{k=1}^{\infty} \|y_k\|$ converges in $\mathbf{R}$, it follows that $\sum_{k=1}^{\infty} y_k$ converges in X. For each $\ell \in \mathbf{Z}^+$, let s_ℓ be the ℓ^{th} partial sum

$$s_\ell = \sum_{k=1}^{\ell} y_k = \sum_{k=1}^{\ell} (x_{n_{k+1}} - x_{n_k}) = x_{n_{\ell+1}} - x_{n_1}$$

and note that $x_{n_\ell} = s_{\ell-1} + x_{n_1}$ for $\ell \geq 2$. Since the series $\sum_{k=1}^{\infty} y_k$ converges in X, its sequence of partial sums $(s_\ell)_{\ell\geq 1}$ converges in X, and hence the sequence $\big(x_{n_\ell}\big)_{\ell\geq 1}$ converges in X. Since $(x_n)_{n\geq 1}$ is a Cauchy sequence, and the subsequence $(x_{n_\ell})_{\ell\geq 1}$ converges, it follows that $(x_n)_{n\geq 1}$ converges by Theorem 4.11.

8: Let X be a metric space.

(a) Show that X is complete if and only if every decreasing sequence of closed balls

$$\overline{B}(a_1,r_1)\supset\overline{B}(a_2,r_2)\supset\overline{B}(a_3,r_3)\supset\cdots$$

in X with $r_n\to 0$ has a non-empty intersection.

Solution: Suppose that X is complete. Let $\overline{B}(a_1,r_1)\supset\overline{B}(a_2,r_2)\supset\overline{B}(a_3,r_3)\supset\cdots$ be a decreasing sequence of balls in X with $r_n\to 0$. We claim that $\langle a_n\rangle$ is Cauchy. Let $\epsilon>0$. Choose $N\in\mathbf{N}$ so that $r_N\leq\frac{\epsilon}{2}$. For $n,m\in\mathbf{N}$ with $n,m\geq N$ we have $a_n,a_m\in B(a_N,r_N)$ so that $d(a_n,a_m)\leq d(a_n,a_N)+d(a_N,a_m)<2r_N\leq\epsilon$, and so $\langle a_n\rangle$ is Cauchy as claimed. Since X is complete, $\langle a_n\rangle$ converges in X. Let $a=\lim\limits_{n\to\infty}a_n$. Note that $a\in\bigcap\limits_{n=1}^{\infty}\overline{B}(a_n,r_n)$ since for each $N\in\mathbf{N}$, the sequence $\langle a_n|n\geq N\rangle$ lies in $\overline{B}(a_N,r_N)$ which is closed in X and hence complete, and so $a\lim\limits_{n\to\infty}a_n\in\overline{B}(a_N,r_N)$.

Conversely, suppose that every decreasing sequence of balls $\overline{B}(a_1,r_1)\supset\overline{B}(a_2,r_2)\supset\overline{B}(a_3,r_3)\supset\cdots$ with $r_n\to 0$ has non-empty intersection. Let $\langle a_n\rangle$ be a Cauchy sequence in X. Choose $n_0\geq 0$ so that for all $n,m\in\mathbf{N}$ we have $n,m\geq n_0\Longrightarrow d(a_n,a_m)<\frac{1}{2}$. Having chosen $n_0<n_1<\cdots<n_{k-1}$, choose $n_k>n_{k-1}$ so that for all $n,m\in\mathbf{N}$ we have $n,m\geq n_k\Longrightarrow d(a_n,a_m)<\frac{1}{2^{k+1}}$. Note that $\overline{B}\left(a_{n_k},\frac{1}{2^k}\right)\subset\overline{B}\left(a_{n_{k-1}},\frac{1}{2^{k-1}}\right)$ since

$$d\big(x,a_{n_k}\big)\leq\tfrac{1}{2^k}\Longrightarrow d\big(x,a_{n_{k-1}}\big)\leq d\big(x,a_{n_{k-1}}\big)+d\big(a_{n_{k1}},a_{n_{k-1}}\big)<\tfrac{1}{2^k}+\tfrac{1}{2^k}=\tfrac{1}{2^{k-1}}\,.$$

Since this decreasing sequence of closed balls has non-empty intersection, we can choose $a\in\bigcap\limits_{n=1}^{\infty}\overline{B}\big(a_{n_k},\frac{1}{2^k}\big)$. Note that $a_{n_k}\to a$ in X since given $\epsilon>0$ we can choose $K\in\mathbf{N}$ so that $\frac{1}{2^{k-1}}<\epsilon$ and then for $k\geq K$ we have $d\big(a_{n_k},a_{n_K}\big)<\frac{1}{2^{K+1}}$ by the choice of n_K, and we have $a\in\overline{B}\big(a_{n_K},\frac{1}{2^K}\big)$ so that $d\big(a,a_{n_K}\big)\leq\frac{1}{2^K}$, and so $d\big(a_{n_k},a\big)\leq d\big(a_{n_k},a_{n_K}\big)+d\big(a_{n_K},a\big)<\frac{1}{2^{K+1}}+\frac{1}{2^K}<\frac{1}{2^{K-1}}<\epsilon$. Finally note that since $\langle a_n\rangle$ is Cauchy and has a convergent subsequence, $\langle a_n\rangle$ converges.

(b) Show that the requirement in part (a) that $r_n\to 0$ is necessary.

Solution: Let $X=\left\{\frac{1}{2^n}\middle|n\in\mathbf{N}\right\}$. Define $d:X\times X\to[0,\infty)$ by

$$d(x,y)=\begin{cases}0 & \text{, if } x=y\\ 1+|x-y| & \text{, if } x\neq y\,.\end{cases}$$

Then d is clearly positive definite and symmetric, and by considering that cases $x=y=z$, $x=y\neq z$, $x=z\neq y$, $y=z\neq x$ and x,y,z all distinct, we see that d satisfies the triangle equality, so d is a metric on X. Under this metric, X is complete since if a sequence in X is Cauchy, then it must be eventually constant, so it converges. But if we take $a_n=\frac{1}{2^n}$ and $r_n=1+\frac{1}{2^n}$, then we have $\overline{B}(a_n,r_n)=\left\{\frac{1}{2^k}\middle|k\geq n-1\right\}$, so $\overline{B}(a_1,r_1)\supset\overline{B}(a_2,r_2)\supset\overline{B}(a_3,r_3)\supset\cdots$ but $\bigcap_{n=1}^{\infty}\overline{B}(a_n,r_n)=\emptyset$.