AMATH/PMATH 331 Real Analysis, Solutions to the Problems for Chapter 5

1: Let X and Y be metric spaces.

(a) Let A and B be closed sets in X with $X = A \cup B$, let $f : A \to Y$ and $g : B \to Y$ be continuous with $f(x) = g(x)$ for all $x \in A \cap B$, and define $h : X \to Y$ by

$$h(x) = \begin{cases} f(x) \text{ , for } x \in A, \\ g(x) \text{ , for } x \in B\,. \end{cases}$$

Show that h is continuous.

Solution: Let $C \subseteq Y$ be closed. Then

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)\,,$$

is closed (since it is the union of two closed sets).

(b) Let A be a dense subset of X and let $f, g : X \to Y$ be continuous maps with $f(x) = g(x)$ for all $x \in A$. Show that $f(x) = g(x)$ for all $x \in X$.

Solution: Let $B = \{x \in X \big| f(x) = g(x)\}$. Note that $A \subseteq B$. We claim that B is closed. Let $a \in B^c$ so that $f(a) \neq g(a)$. Let $r = \frac{1}{2}\, d_Y\big(f(a), g(a))\big)$ so that $B_Y\big(f(a), r\big) \cap B_Y\big(g(a), r\big) = \emptyset$. Since f and g are continuous, the set $U = f^{-1}\big(B_Y\big(f(a), r\big)\big) \cap g^{-1}\big(B_Y\big(g(a), r\big)\big)$ is open, and we have $a \in U$. Choose $s > 0$ so that $B_X(a, s) \subseteq U$. Then for $x \in B_X(a, s)$, we have $f(x) \in B_Y\big(f(a), r\big)$ and $g(x) \in B_Y\big(g(a), r\big)$. Since $B_Y\big(f(a), r\big) \cap B_Y\big(g(a), r\big) = \emptyset$, we see that $f(x) \neq g(x)$, so $x \in B^c$. Thus B^c is open, so B is closed, as claimed. Since B is closed and $A \subseteq B$, we have $\overline{A} \subseteq B$. But A is dense in X, so $\overline{A} = X$, and so we have $X \subseteq B$. Thus $B = X$, as required.

2: Let X and Y be metric spaces, and let $f : X \to Y$.

(a) Show that f is continuous if and only if for every $B \subseteq Y$ we have $f^{-1}(B^\circ) \subseteq f^{-1}(B)^\circ$.

Solution: Suppose that f is continuous. Since A° is open and f is continuous, $f^{-1}\big(A^\circ\big)$ is open. Since $A^\circ \subseteq A$, we have $f^{-1}\big(A^\circ\big) \subseteq f^{-1}(A)$. Since $f^{-1}\big(A^\circ\big)$ is open and $f^{-1}\big(A^\circ\big) \subseteq f^{-1}(A)$, we have $f^{-1}\big(A^\circ\big) \subseteq f^{-1}(A)^\circ$.

Conversely, suppose that for every $A \subseteq Y$ we have $f^{-1}\big(A^\circ\big) \subseteq f^{-1}(A)^\circ$. Let $U \subseteq Y$ be open. Then $U^\circ = U$, so $f^{-1}(U) = f^{-1}\big(U^\circ\big) \subseteq f^{-1}(U)^\circ$. Since $f^{-1}(U) \subseteq f^{-1}(U)^\circ$ and of course $f^{-1}(U)^\circ \subseteq f^{-1}(U)$, we have that $f^{-1}(U) = f^{-1}(U)^\circ$, so $f^{-1}(U)$ is open. Thus f is continuous.

(b) Show that f is continuous if and only if for every $A \subseteq X$ we have $f\big(\overline{A}\big) \subseteq \overline{f(A)}$.

Solution: Suppose that f is continuous. Let $A \subseteq X$. Let $b \in f\big(\overline{A}\big)$, say $b = f(a)$ where $a \in \overline{A}$. We must show that $b \in \overline{f(A)}$. Let $r > 0$. Since $B_Y(b, r)$ is open and f is continuous, $f^{-1}\big(B_Y(b, r)\big)$ is open, so we can choose $s > 0$ so that $B_X(a, s) \subseteq f^{-1}\big(B_Y(b, r)\big)$. Since $a \in \overline{A}$, we have $B_X(a, s) \cap A \neq \emptyset$, so we can choose a point $c \in B_X(a, s) \cap A$. Since $c \in B_X(a, s) \subseteq f^{-1}\big(B_Y(b, r)\big)$ we have $f(c) \in B_Y(b, r)$, and since $c \in A$ we have $f(c) \in f(A)$, and so $f(c) \in B_Y(b, r) \cap f(A)$. Thus $B_Y(b, r) \cap f(A) \neq \emptyset$ and so $b \in \overline{f(A)}$, as required.

Conversely, suppose that for every $A \subseteq X$ we have $f\big(\overline{A}\big) \subseteq \overline{f(A)}$. Let $B \subseteq Y$ be closed. We claim that $f^{-1}(B)$ is closed. Let $A = f^{-1}(B)$. Note that $f(A) \subseteq B$. Let $x \in \overline{A}$. Then $f(x) \in f\big(\overline{A}\big) \subseteq \overline{f(A)} \subseteq \overline{B} = B$ and so $x \in f^{-1}(B) = A$. Thus $\overline{A} \subseteq A$. Of course we also have $A \subseteq \overline{A}$, so $A = \overline{A}$, and so A is closed, as claimed. Thus f is continuous.

3: Determine whether $G : \big(\mathcal{C}[0,1], d_1\big) \to \big(\mathbf{R}, d_2\big)$ given by $G(f) = f(0)$ is continuous.

Solution: We claim that G is not continuous. Since G is linear, it suffices to show that $G\big(\overline{B}_1(0,1)\big)$ is not bounded. For $n \geq 1$, define $f_n : [0,1] \to \mathbf{R}$ by

$$f_n(x) = \begin{cases} 2n - 2n^2 x \ , & 0 \leq x \leq \frac{1}{n} \\ 0 \ , & \frac{1}{n} \leq x \leq 1\,. \end{cases}$$

Then $\|f_n\|_1 = \displaystyle\int_0^1 |f_n(x)|\,dx = 1$, so $f_n \in \overline{B}_1(0,1)$, but $G(f_n) = f_n(0) = 2n$, so $G\big(\overline{B}_1(0,1)\big)$ is unbounded.

(b) Determine whether $H : \big(\mathcal{C}[0,1], d_2\big) \to (\mathbf{R}, d_2)$ given by $H(f) = \displaystyle\int_0^1 f(x)\,dx$ is continuous.

Solution: We claim that H is continuous. Since H is linear, it suffices to show that $H\big(\overline{B}_2(0,1)\big)$ is bounded. Let $f \in \overline{B}_2(0,1)$ so we have $\|f\|_2 \leq 1$, that is $\displaystyle\int_0^1 f(x)^2 dx \leq 1$. Then, since $|y| \leq 1 + y^2$ for all $y \in \mathbf{R}$, we have

$$\big|H(f)\big| = \left|\int_0^1 f(x)\,dx\right| \leq \int_0^1 |f(x)|\,dx \leq \int_0^1 1 + f(x)^2\ dx = 1 + \int_0^1 f(x)^2\,dx \leq 2\,,$$

and so $H\big(\overline{B}_2(0,1)\big)$ is bounded, as required.