AMATH/PMATH 331 Real Analysis, Problems for Chapter 5

1: Let X and Y be metric spaces.

(a) Let A and B be closed sets in X with $X = A \cup B$, let $f : A \to Y$ and $g : B \to Y$ be continuous with $f(x) = g(x)$ for all $x \in A \cap B$, and define $h : X \to Y$ by

$$h(x) = \begin{cases} f(x) \ , \text{ for } x \in A, \\ g(x) \ , \text{ for } x \in B \, . \end{cases}$$

Show that h is continuous.

(b) Let A be a dense subset of X and let $f, g : X \to Y$ be continuous maps with $f(x) = g(x)$ for all $x \in A$. Show that $f(x) = g(x)$ for all $x \in X$.

2: Let X and Y be metric spaces, and let $f : X \to Y$.

(a) Show that f is continuous if and only if for every $B \subseteq Y$ we have $f^{-1}(B^\circ) \subseteq f^{-1}(B)^\circ$.

(b) Show that f is continuous if and only if for every $A \subseteq X$ we have $f\big(\overline{A}\big) \subseteq \overline{f(A)}$.

3: (a) Determine whether $G : \big(\mathcal{C}[0,1], d_1\big) \to \big(\mathbf{R}, d_2\big)$ given by $G(f) = f(0)$ is continuous.

(b) Determine whether $H : \big(\mathcal{C}[0,1], d_2\big) \to (\mathbf{R}, d_2)$ given by $H(f) = \displaystyle\int_0^1 f(x)\, dx$ is continuous.