AMATH/PMATH 331 Real Analysis, Solutions to the Problems for Chapter 4

1: Determine which of the following functions $d : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are metrics on $\mathbf{R}$.

(a) $d(x, y) = (x - y)^2$

Solution: This is not a metric on $\mathbf{R}$ since it does not satisfy the triangle inequality. For example, if $x = 0$, $y = 1$ and $z = 2$ then $d(x, y) + d(y, z) = 1 + 1 = 2 < 4 = d(x, z)$.

(b) $d(x, y) = \sqrt{|x - y|}$

Solution: This is a metric on $\mathbf{R}$. It is clearly positive definite and symmetric, and for $x, y, z \in \mathbf{R}$ we have

$$\begin{aligned} d(x,z) &= \sqrt{|x-z|} \\ &\le \sqrt{|x-y| + |y-z|} \text{ , by the triangle inequality in } \mathbf{R} \\ &\le \sqrt{|x-y|} + \sqrt{|y-z|} \text{ , by the triangle inequality in } \mathbf{R}^2 \\ &= d(x,y) + d(y,z) \,. \end{aligned}$$

(c) $d(x, y) = |x^2 - y^2|$

Solution: This is not a metric on $\mathbf{R}$ since it is not positive definite. For example, if $x = 1$ and $y = -1$ then we have $d(x, y) = 0$ but $x \neq y$.

(d) $d(x, y) = \dfrac{|x - y|}{1 + |x - y|}$

Solution: This is a metric. More generally, if d_1 is any metric on a set X, and if $F : [0, \infty) \to [0, \infty)$ is any function which satisfies

(1) $F(x) \ge 0$ for all $x \ge 0$, with $F(x) = 0 \iff x = 0$,
(2) $F(x) \le F(y)$ for all $x \le y \in \mathbf{R}$, and
(3) $F(x + y) \le F(x) + F(y)$ for all $x, y \ge 0$,

then the map $d_2(x, y) = F\big(d_1(x, y)\big)$ is also a metric. Indeed property (1) ensures that d_2 is positive-definite, we need no requirement on F to ensure that d_2 is symmetric, and properties (2) and (3) ensure that d_2 satisfies the triangle inequality, since for $x, y, z \in \mathbf{R}$ we have

$$\begin{aligned} d_2(x,z) &= F\big(d_1(x,z)\big) \le F\big(d_1(x,y) + d_1(y,z)\big) \text{ , using propery (2)} \\ &\le F\big(d_1(x,y)\big) + F\big(d_1(y,z)\big) \text{ , using property (3)} \\ &= d_2(x,y) + d_2(y,z) \,. \end{aligned}$$

We note that if $F(0) = 0$, and $F'(t) > 0$ for all $t \ge 0$, then F satisfies properties (1) and (2), and if in addition $F''(t) < 0$ for all $t \ge 0$, then F also satisfies property (3). Indeed, say $0 \le x \le y \le x + y$. Using the Mean Value Theorem, choose a with $0 \le a \le x$ so that $F(x) - F(0) = F'(a)(x - 0)$, that is $F(x) = F'(a)\, x$, and choose b with $y \le b \le x + y$ so that $F(x+y) - F(y) = F'(b)(x+y-y)$, that is $F(x+y) - F(y) = F'(b)\, x$. Since $F''(t) < 0$ for all $t \ge 0$, $F'(t)$ is decreasing, so

$$a \le b \Longrightarrow F'(b) \ge F'(a) \Longrightarrow F'(b)\, x \ge F'(a)\, x \Longrightarrow F(x+y) - F(y) \ge F(x) \,.$$

When d_1 is the standard metric on $\mathbf{R}$ and $F(t) = \dfrac{t}{1+t}$, we have $F(0) = 0$, $F'(t) = \dfrac{1}{(1+t)^2} > 0$ for all $t \ge 0$ and $F''(t) = \dfrac{-2}{(1+t)^3} < 0$ for all $t \ge 0$, and we have $d = d_2$.

2: (a) Let $S=\left\{(x,y)\in\mathbf{R}^2\big|y>x^2\right\}$. Prove, from the definition of an open set, that S is open in $\mathbf{R}^2$.

Solution: Let $(a,b)\in S$ so we have $b>a^2$ and hence $\sqrt{b}>|a|$. Let $r=\min\left(\frac{b-a^2}{2}\,,\,\frac{\sqrt{b}-|a|}{2}\right)$. We claim that $B\big((a,b),r\big)\subseteq S$. Let $(x,y)\in B\big((a,b),r\big)$. Note that

$$|x-a|\leq\sqrt{(x-a)^2+(y-b)^2}=d\big((a,b),(x,y)\big)<r\leq\tfrac{\sqrt{b}-|a|}{2}$$

and similarly

$$|y-b|<r\leq\tfrac{b-a^2}{2}.$$

It follows that $|x|-|a|\leq|x-a|<\frac{\sqrt{b}-|a|}{2}$ so that $|x|\leq\frac{\sqrt{b}+|a|}{2}$ and that $b-y\leq|y-b|<\frac{b-a^2}{2}$ so that $y>\frac{b+a^2}{2}$. Note that $0\leq\big(\sqrt{b}-|a|\big)^2=b+a^2-2|a|\sqrt{b}$ so we have $2|a|\sqrt{b}\leq b+a^2$. It follows that

$$x^2<\left(\tfrac{\sqrt{b}+|a|}{2}\right)^2=\tfrac{b+a^2+2|a|\sqrt{b}}{4}\leq\tfrac{b+a^2}{2}<y.$$

Since $y>x^2$ we have $(x,y)\in S$. This shows that $B\big((a,b),r\big)\subseteq S$, as claimed, and so S is open.

(b) Define $f:\mathbf{R}\to\mathbf{R}^2$ by $f(t)=\left(\frac{2t}{t^2+1},\frac{t^2-1}{t^2+1}\right)$. Show that Range$(f)$ is not closed in $\mathbf{R}^2$.

Solution: To solve this problem, it helps to draw a picture of Range$(f)\subseteq\mathbf{R}^2$. By plotting points, you will see that Range(f) looks like the unit circle centred at $(0,0)$ with the point $(0,1)$ removed and, if you wish, you can show that this is indeed the case. Let $S=$ Range(f) and let $a=(0,1)$. Let $x(t)=\frac{2t}{t^2+1}$ and $y(t)=\frac{t^2-1}{t^2+1}$ so that $f(t)=\big(x(t),y(t)\big)$. We claim that $a\in\overline{S}$ but $a\notin S$. It is clear that $a\notin S$ because to get $f(t)=a$ we need $x(t)=0$ and $y(t)=1$, but to get $x(t)=\frac{2t}{t^2+1}=0$ we must choose $t=0$, but when $t=0$ we have $y(t)=\frac{t^2-1}{t^2+1}=-1\neq1$. To show that $a\in\overline{S}$, we shall show that for all $r>0$ we have $B(a,r)\cap S\neq\emptyset$. Let $r>0$. Since $\lim\limits_{t\to\infty}x(t)=0$ and $\lim\limits_{t\to\infty}y(t)=1$ we can choose $t\in\mathbf{R}$ so that $\big|x(t)-0\big|<\frac{r}{2}$ and $\big|y(t)-1\big|<\frac{r}{2}$. Then we have

$$\big|f(t)-a\big|=\big|(x(t),y(t))-(0,1)\big|=\big|\big(x(t)\,,\,y(t)-1\big)\big|\leq|x(t)|+|y(t)-1|<\tfrac{r}{2}+\tfrac{r}{2}=r$$

and so $f(t)\in B(a,r)\cap S$. This shows that for all $r>0$ we have $B(a,r)\cap S\neq\emptyset$, and so $a\in\overline{S}$. Since $a\in\overline{S}$ but $a\notin S$ we see that $S\neq\overline{S}$ and so S is not closed.

3: Determine which of the following statements are true for every metric space (X,d) and every $A\subseteq X$.

(a) $\overline{B(a,r)}=\overline{B}(a,r)$ for every $a\in X$ and every $r>0$.

Solution: This is false. For example, let $X=\mathbf{Z}$ with the standard metric. Then $B(0,1)=\{0\}=\overline{B(0,1)}$, but $\overline{B}(0,1)=\{-1,0,1\}$.

(b) $\left(\overline{A}\right)^c=(A^c)^\circ$.

Solution: This is true. Indeed

$$\begin{aligned}\left(\overline{A}\right)^c&=\left(\bigcap\left\{K\subseteq X\middle|K\text{ closed},A\subseteq K\right\}\right)^c\\&=\bigcup\left\{K^c\subseteq X\middle|K\text{ closed},A\subseteq K\right\}\\&=\bigcup\left\{U\subseteq X\middle|U^c\text{ closed},A\subseteq U^c\right\}\\&=\bigcup\left\{U\subseteq X\middle|U\text{ open},U\subseteq A^c\right\}=(A^c)^\circ\ .\end{aligned}$$

(c) If $A=A^\circ$ then $A=\left(\overline{A}\right)^\circ$.

Solution: This is false. For example, let $X=\mathbf{R}^n$ with the standard metric, and let $A=B^*(0,1)$. Then $\overline{A}=\overline{B}(0,1)$ and $\left(\overline{A}\right)^\circ=B(0,1)$.

(d) If $A=\overline{A}$ then $\partial(\partial A)=\partial A$.

Solution: This is true, as we now prove. Note first that, for any set A, we have $\partial A=\overline{A}\setminus A^\circ=\overline{A}\cap(A^\circ)^c$, which is closed, since it is the intersection of two closed sets. Now suppose that A is closed. We claim that $(\partial A)^\circ=\emptyset$. Indeed, if we had $a\in(\partial A)^\circ$, then we could choose $r>0$ so that $B(a,r)\subseteq\partial A$, but then we would have $B(a,r)\subseteq\partial A\subseteq\overline{A}=A$ so that $a\in A^\circ$, and this is not possible since $a\in A^\circ\Longrightarrow a\notin\partial A\Longrightarrow a\notin(\partial A)^\circ$. Since ∂A is closed with $(\partial A)^\circ=\emptyset$, we have $\partial(\partial A)=\overline{\partial A}\setminus(\partial A)^\circ=\partial A\setminus\emptyset=\partial A$.

4: (a) Show that there is no inner product on $\mathbf{R}^2$ which induces the 1-norm $\|\ \|_1$.

Solution: If there were such an inner product, say $\langle\ ,\ \rangle$, then by the polarization identity we would have

$$\begin{aligned}\langle(1,0),(0,1)\rangle&=\tfrac12\left(\left\|(1,1)\right\|_1{}^2-\left\|(1,0)\right\|_1{}^2-\left\|(0,1)\right\|_1{}^2\right)=\tfrac12(4-1-1)=1\ ,\text{ and}\\\langle-(1,0),(0,1)\rangle&=\tfrac12\left(\left\|(-1,1)\right\|_1{}^2-\left\|(-1,0)\right\|_1{}^2-\left\|(0,1)\right\|_1{}^2\right)=\tfrac12(4-1-1)=1\,,\end{aligned}$$

but this is not possible since by linearity, we must have $\langle-(1,0),(0,1)\rangle=-\langle(1,0),(0,1)\rangle$.

(b) Let $T=\left\{U\subseteq\mathbf{R}\,\middle|\,U=\emptyset\text{ or }\mathbf{R}\setminus U\text{ is finite}\right\}$. Show that T is a topology on $\mathbf{R}$ which is not induced by any metric on $\mathbf{R}$ (T is called the *cofinite topology* on $\mathbf{R}$).

Solution: First we show that T is a topology on $\mathbf{R}$. Clearly, we have $\emptyset\in T$ and $\mathbf{R}\in T$ (since $\mathbf{R}^c=\emptyset$, which is finite). Suppose that $U_\alpha\in T$ for each $\alpha\in A$. If every $U_\alpha=\emptyset$ then $\bigcup\limits_{\alpha\in A}U_\alpha=\emptyset$, so $\bigcup\limits_{\alpha\in A}U_\alpha\in T$. If $U_\beta\neq\emptyset$ for some $\beta\in A$, then since $U_\beta\subseteq\bigcup\limits_{\alpha\in A}U_\alpha$, we have $\big(\bigcup\limits_{\alpha\in A}U_\alpha\big)^c\subseteq{U_\beta}^c$, which is finite, so $\bigcup\limits_{\alpha\in A}U_\alpha\in T$. Suppose that $U_k\in T$ for each $k=0,1,\cdots,n$. If some $U_k=\emptyset$ then $\bigcap\limits_{k=0}^nU_k=\emptyset$, so $\bigcap\limits_{k=0}^nU_k\in T$. If no $U_k=\emptyset$ then each ${U_k}^c$ is finite, so $\big(\bigcap\limits_{k=0}^nU_k\big)^c=\bigcup\limits_{k=0}^n{U_k}^c$, which is a finite union of finite sets, and hence finite, so $\bigcap\limits_{k=0}^nU_k\in T$.

Next we show that T cannot be induced by any metric. Let d be any metric on $\mathbf{R}$. Let $r=\frac12\,d(0,1)$. Note that $B(0,r)\cap B(1,r)=\emptyset$ since if we had $x\in B(0,r)\cap B(1,r)$ then we would have $d(0,x)<r$ and $d(1,x)<r$ and so $2r=d(0,1)\le d(0,x)+d(x,1)<r+r=2r$, which is not possible. Thus in the topology which is induced by any metric, there exist two disjoint non-empty sets. On the other hand, in the cofinite topology T on $\mathbf{R}$, given any two non-empty sets $U_1,U_2\in T$, as shown in the previous paragraph we have that $(U_1\cap U_2)^c$ is finite, and so $U_1\cap U_2\neq\emptyset$.

5: (a) Show that ℓ_1 is neither open nor closed in the metric space (ℓ_∞, d_∞).

Solution: To see that ℓ_1 is not closed in (ℓ_∞, d_∞), for each $n \in \mathbf{N}$ let $a_n = (a_{nk} | k \in \mathbf{N})$ be the sequence given by $a_{nk} = \frac{1}{k+1}$ for $0 \leq k < n$ and $a_{nk} = 0$ for $k \geq n$. Then each $a_n \in k_0 \subseteq \ell_1$, but we have $\lim\limits_{n\to\infty} a_n = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right) \notin \ell_1$.

To see that ℓ_1 is not open, and indeed to see that ${\ell_1}^\circ = \emptyset$ in (ℓ_∞, d_∞), note that given $a = (a_n) \in \ell_1$ and $0 < r \in \mathbf{R}$, we can choose $b = (b_n)$ to be the sequence given by $b_n = a_n + \frac{r}{2}$. Then we have $b \in B_\infty(a, r)$ but $b \notin \ell_1$.

(b) Determine whether every set $U \subseteq \ell_1$ which is open in (ℓ_1, d_2) is also open in (ℓ_1, d_1).

Solution: We show that this is indeed the case. Let $a = (a_n)_{n\geq 1}$ and $x = (x_n)_{n\geq 1}$ be in ℓ_2. For all $N \in \mathbf{Z}^+$, the Triangle Inequality gives

$$\sqrt{\sum_{n=1}^{N} (a_n - x_n)^2} \leq \sum_{n=1}^{N} |a_n - x_n| \,.$$

Taking the limit as $N \to \infty$ we obtain

$$d_2(a, x) = \sqrt{\sum_{n=1}^{\infty} (a_n - x_n)^2} \leq \sum_{n=1}^{\infty} |a_n - x_n| = d_1(a, x) \,.$$

It follows that for all $0 < r \in \mathbf{R}$ we have $B_1(a, r) \subseteq B_2(a, r)$ since

$$x \in B_1(a, r) \Longrightarrow d_1(a, x) < r \Longrightarrow d_2(a, x) \leq d_1(a, x) < r \Longrightarrow x \in B_2(a, r) \,.$$

Now suppose that $U \subseteq \ell_1$ is open in (ℓ_1, d_2). Let $a \in U$. Choose $0 < r \in \mathbf{R}$ so that $B_2(a, r) \subseteq U$. Then $B_1(a, r) \subseteq B_2(a, r) \subseteq U$. Thus U is also open in (ℓ_1, d_1).

(c) Determine whether every set $U \subseteq \ell_1$ which is open in (ℓ_1, d_1) is also open in (ℓ_1, d_2).

Solution: We shall show that for $a \in \ell_1$ and $0 < r \in \mathbf{R}$, the ball $B_1(a, r)$, which is open in (l_1, d_1), is not open in (l_1, d_2). Let $a = (a_n)_{n\geq 1} \in \ell_1$ and let $0 < r \in \mathbf{R}$. We claim that $a \notin B_1(a, r)^\circ$ in (ℓ_1, d_2). We must show that for every $0 < s \in \mathbf{R}$, $B_2(a, s) \not\subseteq B_1(a, r)$. Let $s > 0$. Recall that $\sum \frac{1}{n^2}$ converges but $\sum \frac{1}{n}$ diverges. Let

$$p = \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}}$$

$\left(\text{in fact, } p = \frac{\pi}{\sqrt{6}}\right)$. Then we have $\sum\limits_{n=1}^{\infty} \frac{s^2/p^2}{n^2} = s^2$ and $\sum\limits_{n=1}^{\infty} \frac{s/p}{n} = \infty$. Choose $N \in \mathbf{Z}^+$ so that $\sum\limits_{n=1}^{N} \frac{s/p}{n} > r$. Let $b = (b_n)_{n\geq 1}$ be the sequence given by $b_n = \frac{s/p}{n}$ for $1 \leq n \leq N$ and $b_n = 0$ for $n > N$. Note that $b \in \mathbf{R}^\infty \subseteq \ell_1$ so $a + b \in \ell_1$. We have $a + b \in B_2(a, s)$ because

$$d_2(a, a + b) = \sqrt{\sum_{n=1}^{N} \frac{s^2/p^2}{n^2}} < \sqrt{\sum_{n=1}^{\infty} \frac{s^2/p^2}{n^2}} = s \,,$$

but $a + b \notin B_1(a, r)$ because

$$d_1(a, a + b) = \sum_{n=1}^{N} \frac{s/p}{n} > r \,.$$

Thus $B_2(a, r) \not\subseteq B_1(a, r)$, as required. We remark that a minor modification of the above argument can be used to show that $B_1(a, r)^\circ = \emptyset$ in (ℓ_1, d_2).