AMATH/PMATH 331 Real Analysis, Problems for Chapter 4

1: Determine which of the following functions $d : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are metrics on $\mathbf{R}$.

(a) $d(x,y) = (x-y)^2$

(b) $d(x,y) = \sqrt{|x-y|}$

(c) $d(x,y) = |x^2 - y^2|$

(d) $d(x,y) = \dfrac{|x-y|}{1+|x-y|}$

2: (a) Let $S = \left\{(x,y) \in \mathbf{R}^2 \middle| y > x^2\right\}$. Prove, from the definition of an open set, that S is open in $\mathbf{R}^2$.

(b) Define $f : \mathbf{R} \to \mathbf{R}^2$ by $f(t) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$. Show that Range$(f)$ is not closed in $\mathbf{R}^2$.

3: Determine which of the following statements are true for every metric space (X,d) and every $A \subseteq X$.

(a) $\overline{B(a,r)} = \overline{B}(a,r)$ for every $a \in X$ and every $r > 0$.

(b) $\left(\overline{A}\right)^c = (A^c)^\circ$.

(c) If $A = A^\circ$ then $A = \left(\overline{A}\right)^\circ$.

(d) If $A = \overline{A}$ then $\partial(\partial A) = \partial A$.

4: (a) Show that there is no inner product on $\mathbf{R}^2$ which induces the 1-norm $\|\ \|_1$.

(b) Let $T = \left\{U \subseteq \mathbf{R} \middle| U = \emptyset \text{ or } \mathbf{R} \setminus U \text{ is finite}\right\}$. Show that T is a topology on $\mathbf{R}$ which is not induced by any metric on $\mathbf{R}$ (T is called the *cofinite topology* on $\mathbf{R}$).

5: (a) Show that ℓ_1 is neither open nor closed in the metric space (ℓ_∞, d_∞).

(b) Determine whether every set $U \subseteq \ell_1$ which is open in (ℓ_1, d_2) is also open in (ℓ_1, d_1).

(c) Determine whether every set $U \subseteq \ell_1$ which is open in (ℓ_1, d_1) is also open in (ℓ_1, d_2).