AMATH/PMATH 331 Real Analysis, Solutions to the Problems for Chapter 3

1: For each of the following sequences of functions (f_n), find the set A of points $x \in \mathbf{R}$ for which $\big(f_n(x)\big)$ converges, and find the (pointwise) limit function $f(x) = \lim\limits_{n\to\infty} f_n(x)$ for $x \in A$.

(a) $f_n(x) = (\sin x)^n$

Solution: If $x = \frac{\pi}{2} + 2\pi k$ for some $k \in \mathbf{Z}$ then $\sin x = 1$ and so $f_n(x) = 1$ for all n, and so $\lim_{n\to\infty} f_n(x) = 1$. If $x = -\frac{\pi}{2} + 2\pi k$ for some $k \in \mathbf{Z}$ then $\sin x = -1$ so $f_n(x) = (-1)^n$ and so $\lim\limits_{n\to\infty} f_n(x)$ does not exist. If $x \neq \frac{\pi}{2} + \pi k$ for any $k \in \mathbf{Z}$ then $|\sin x| < 1$ so $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} (\sin x)^n = 0$. Thus the limit function is

$$f(x) = \begin{cases} 0 \ , \text{ if } x \neq \frac{\pi}{2} + \pi k \text{ for any } k \in \mathbf{Z} \\ 1 \ , \text{ if } x = \frac{\pi}{2} + 2\pi k \text{ for some } k \in \mathbf{Z} \end{cases}$$

and $f(x)$ is not defined when $x = -\frac{\pi}{2} + 2\pi k$ for some $k \in \mathbf{Z}$.

(b) $f_n(x) = (\sin x)^{1/(2n+1)}$

Solution: If $x = \pi k$ for some $k \in \mathbf{Z}$ then $\sin x = 0$ so $f_n(x) = 0$ and so $\lim\limits_{n\to\infty} f_n(x) = 0$. If $x \in \big(2\pi k, \pi(2k+1)\big)$ for some $k \in \mathbf{Z}$ then $0 < \sin x \leq 1$ and so $\lim\limits_{n\to\infty} f_n(x) = 1$. If $x \in \big(\pi(2k-1), 2\pi k\big)$ for some $k \in \mathbf{Z}$ then $-1 \leq \sin x < 0$ and so $\lim\limits_{n\to\infty} f_n(x) = -1$. Thus the limit function is

$$f(x) = \begin{cases} 0 \ , \text{ if } x = \pi k \\ 1 \ , \text{ if } x \in \big(2\pi k, \pi(2k+1)\big) \\ -1 \ , \text{ if } x \in \big(\pi(2k-1), 2\pi k\big) \end{cases}$$

2: (a) Find $\int_0^1 \lim_{n\to\infty} nx(1-x^2)^n\,dx$ and $\lim_{n\to\infty}\int_0^1 nx(1-x^2)^n\,dx$.

Solution: Let $x\in[0,1]$. If $x=0$ or $x=1$ then $nx(1-x^2)^n=0$ for all n and so $\lim_{n\to\infty} nx(1-x^2)^n=0$. If $x\in(0,1)$ then $0<(1-x^2)<1$, so the series $\sum nx(1-x^2)^n$ converges by the Ratio Test and so $\lim_{n\to\infty} nx(1-x^2)^n=0$ by the Divergence Test. Thus $\int_0^1 \lim_{n\to\infty} nx(1-x^2)^n\,dx=\int_0^1 0\,dx=0$. On the other hand, using the substitution $u=1-x^2$ so $du=-2x\,dx$ we have

$$\int_0^1 nx(1-x^2)^n\,dx=\int_1^0 -\tfrac{1}{2}n\,u^2\,du=\left[\frac{-n\,u^{n+1}}{2(n+1)}\right]_1^0=\frac{n}{2(n+1)}\,,$$

and so we have $\lim_{n\to\infty}\int_0^1 nx(1-x^2)^n\,dx=\frac{1}{2}$.

(b) Find $\int_1^4 \lim_{n\to\infty}\frac{\tan^{-1}(nx)}{x}\,dx$ and $\lim_{n\to\infty}\int_1^4 \frac{\tan^{-1}(nx)}{x}\,dx$.

Solution: Let $x\in[1,4]$. Then $\lim_{n\to\infty}\frac{\tan^{-1}(nx)}{x}=\frac{\pi}{2x}$ and so

$$\int_1^4 \lim_{n\to\infty}\frac{\tan^{-1}(nx)}{x}\,dx=\int_1^4\frac{\pi}{2x}\,dx=\Big[\tfrac{\pi}{2}\ln x\Big]_1^4=\pi\ln 2\,.$$

We claim that $\left\{\frac{\tan^{-1}(nx)}{x}\right\}\to\frac{\pi}{2x}$ uniformly on $[1,4]$. Indeed, given $\epsilon>0$ we can choose N so that $x\ge N\implies \left|\tan^{-1}x-\frac{\pi}{2}\right|<\epsilon$ for all $x\ge N$. Then for $n\ge N$ and $x\ge 1$ we have

$$\left|\frac{\tan^{-1}(nx)}{x}-\frac{\pi}{2x}\right|=\frac{\left|\tan^{-1}(nx)-\frac{\pi}{2}\right|}{x}<\frac{\epsilon}{x}\le\epsilon\,.$$

Since the convergence is uniform, $\lim_{n\to\infty}\int_1^4\frac{\tan^{-1}(nx)}{x}\,dx=\int_1^4\lim_{n\to\infty}\frac{\tan^{-1}(nx)}{x}\,dx=\pi\ln 2$.

(c) Show that $\sum_{n=0}^\infty\frac{\cos(2^nx)}{1+n^2}$ converges uniformly on $\mathbf{R}$ and find $\int_0^{\pi/4}\sum_{n=0}^\infty\frac{\cos(2^nx)}{1+n^2}\,dx$.

Solution: For all $x\in\mathbf{R}$ we have $\left|\frac{\cos(2^nx)}{1+n^2}\right|\le\frac{1}{1+n^2}<\frac{1}{n^2}$, and $\sum\frac{1}{n^2}$ converges, so $\sum_{n=0}^\infty\frac{\cos(2^nx)}{1+n^2}$ converges uniformly by the Weirstrass M-Test. Since the convergence is uniform,

$$\int_0^{\pi/4}\sum_{n=0}^\infty\frac{\cos(2^nx)}{1+n^2}\,dx=\sum_{n=0}^\infty\int_0^{\pi/4}\frac{\cos(2^nx)}{1+n^2}\,dx=\sum_{n=0}^\infty\left[\frac{1}{2^n}\,\frac{\sin(2^nx)}{1+n^2}\right]_0^{\pi/4}=\tfrac{\sqrt{2}}{2}+\tfrac{1}{4}+0+0+\cdots=\tfrac{\sqrt{2}}{2}+\tfrac{1}{4}.$$

3: Suppose that (f_n) and (g_n) converge uniformly on $A \subseteq \mathbf{R}$.

(a) Show that if f and g are bounded on A then $(f_n g_n)$ converges uniformly on A.

Solution: Let $f = \lim_{n\to\infty} f$ and $g = \lim_{n\to\infty} g_n$, and suppose that f and g are bounded on A, say $|f(x)| \leq M$ and $|g(x)| \leq M$ for all $x \in A$. Choose N_1 so that $n \geq N_1 \Longrightarrow |f_n(x) - f(x)| < 1$ for all $x \in A$. Note that for $n \geq N_1$ we have $|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| \leq M + 1$. Now choose $N \geq N_1$ so that when $n \geq N$ we have $|f_n(x) - f(x)| < \frac{\epsilon}{2M}$ and $|g_n(x) - g(x)| < \frac{\epsilon}{2(M+1)}$ for all $x \in A$. Then when $n \geq N$, for all $x \in A$ we have

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)| \\ &= |f_n(x)||g_n(x) - g(x)| + |f_n(x) - f(x)||g(x)| \\ &\leq (M+1)\tfrac{\epsilon}{2(M+1)} + \tfrac{\epsilon}{2M}\, M = \epsilon\,. \end{aligned}$$

Thus $f_n g_n \to fg$ uniformly on A.

(b) Show that if f and g are not bounded then $(f_n g_n)$ does not necessarily converge uniformly on A.

Solution: Let $A = \mathbf{R}$, let $f(x) = g(x) = x$ and let $f_n(x) = g_n(x) = x + \frac{1}{n}$. Then $f_n(x)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2}$ so $\lim_{n\to\infty} f_n(x)^2 = x^2 = f(x)^2$ for all $x \in \mathbf{R}$, but the convergence is not uniform, since given any positive integer n, when $x \geq n$ we have $|f_n(x)^2 - f(x)^2| = \frac{2x}{n} + \frac{1}{n^2} > 2$.

7: Determine which of the following statements are true for all sequences of functions (f_n).

(a) If (f_n) converges uniformly on (a, b) and pointwise on $[a, b]$ then (f_n) converges uniformly on $[a, b]$.

Solution: This is true. Indeed, suppose that (f_n) converges uniformly in (a, b) and that $\big(f_n(a)\big)$ and $\big(f_n(b)\big)$ both converge. Then given $\epsilon > 0$ we can choose N so that when $l, m \geq N$ we have $|f_l(x) - f_m(x)| < \epsilon$ for all $x \in (0, 1)$, and $|f_l(a) - f_m(a)| < \epsilon$ and $|f_l(b) = f_m(b)| < \epsilon$, and so we have $|f_l(x) - f_m(x)| < \epsilon$ for all $x \in [a, b]$.

(b) If each f_n is continuous on $[a, b]$ and $\sum f_n$ converges uniformly on $[a, b]$ then $\sum M_n$ converges, where $M_n = \max\big\{|f_n(x)|\big|a \leq x \leq b\big\}$.

Solution: This is false. For a counterexample, let

$$f_n(x) = \begin{cases} \frac{1}{n}\sin^2(2^n \pi\, x)\ , \text{ if } \frac{1}{2^n} \leq x \leq \frac{1}{2^{n-1}} \\ 0\ , \text{ otherwise.} \end{cases}$$

Then $M_n = \frac{1}{n}$ so $\sum M_n$ diverges, and yet we claim that $\sum f_n$ converges uniformly on $[0, 1]$. Indeed if we write $S(x) = \sum_{n=1}^{\infty} f_n(x)$ and $S_l(x) = \sum_{n=l}^{\infty} f_n(x)$ then for all $x \in [0, 1]$ we have

$$\big|S_l(x) - S(x)\big| = \sum_{n=l+1}^{\infty} f_n(x) \leq \max\{M_{l+1}, M_{l+2}, \cdots\} = \tfrac{1}{l+1}$$

since for each x, at most one of the terms $f_n(x)$ is non-zero.