AMATH/PMATH 331 Real Analysis, Problems for Chapter 3

1: For each of the following sequences of functions (f_n), find the set A of points $x \in \mathbf{R}$ for which $\big(f_n(x)\big)$ converges, and find the (pointwise) limit function $f(x) = \lim\limits_{n\to\infty} f_n(x)$ for $x \in A$.

(a) $f_n(x) = (\sin x)^n$

(b) $f_n(x) = (\sin x)^{1/(2n+1)}$

2: (a) Find $\displaystyle\int_0^1 \lim_{n\to\infty} nx(1-x^2)^n\,dx$ and $\displaystyle\lim_{n\to\infty}\int_0^1 nx(1-x^2)^n\,dx$.

(b) Find $\displaystyle\int_1^4 \lim_{n\to\infty} \frac{\tan^{-1}(nx)}{x}\,dx$ and $\displaystyle\lim_{n\to\infty}\int_1^4 \frac{\tan^{-1}(nx)}{x}\,dx$.

(c) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(2^n x)}{1+n^2}$ converges uniformly on $\mathbf{R}$ and find $\displaystyle\int_0^{\pi/4} \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{1+n^2}\,dx$.

3: Suppose that (f_n) and (g_n) converge uniformly on $A \subseteq \mathbf{R}$.

(a) Show that if f and g are bounded on A then $(f_n g_n)$ converges uniformly on A.

(b) Show that if f and g are not bounded then $(f_n g_n)$ does not necessarily converge uniformly on A.

4: Determine which of the following statements are true for all sequences of functions (f_n).

(a) If (f_n) converges uniformly on (a,b) and pointwise on $[a,b]$ then (f_n) converges uniformly on $[a,b]$.

(b) If each f_n is continuous on $[a,b]$ and $\sum f_n$ converges uniformly on $[a,b]$ then $\sum M_n$ converges, where $M_n = \max\big\{|f_n(x)|\big|a \le x \le b\big\}$.