AMATH/PMATH 331 Real Analysis, Solutions to the Problems for Chapter 2

1: (a) Let $0 \le a < b$. Let $f(x) = x^2$. From the definition of integrability, show that f is integrable on $[a,b]$ with $\int_a^b f = \frac{1}{3}(b^3 - a^3)$.

Solution: Let $\epsilon > 0$ be arbitrary. Choose $\delta = \frac{\epsilon}{2b(b-a)}$. Let X be any partition of $[a,b]$ with $|X| < \delta$. Let $t_i \in [x_{i-1}, x_i]$ be any sample points. Let $s_i = \sqrt{\frac{1}{3}\left(x_{i-1}{}^2 + x_{i-1}x_i + x_i{}^2\right)} \in [x_{i-1}, x_i]$. Note that $\sum_{i=1}^n f(s_i)\Delta_i x = \sum_{i=1}^n \frac{1}{3}\left(x_{i-1}{}^2 + x_{i-1}x_i + x_i{}^2\right)(x_i - x_{i-1}) = \sum_{i=1}^n \frac{1}{3}(x_i{}^3 - x_{i-1}{}^3) = \frac{1}{3}(b^3 - a^3)$, so

$$\begin{aligned}
\left|\sum_{i=1}^n f(t_i)\Delta_i x - \tfrac{1}{3}(b^3 - a^3)\right| &= \left|\sum_{i=1}^n f(t_i)\Delta_i x - \sum_{i=1}^n f(s_i)\Delta_i x\right| \le \sum_{i=1}^n \big|f(t_i) - f(s_i)\big|\Delta_i x \\
&= \sum_{i=1}^n \big|t_i{}^2 - s_i{}^2\big|\Delta_i x = \sum_{i=1}^n |t_i + s_i||t_i - s_i|\Delta_i x \\
&< \sum_{i=1}^n 2b\,\delta\,\Delta_i x = \epsilon\,.
\end{aligned}$$

(b) Find $\int_0^8 \sqrt[3]{x}\,dx$ by evaluating the limit of a sequence of Riemann sums using the right endpoints of suitable partitions.

Solution: Let $f(x) = \sqrt[3]{x}$ and let $X_n = \{x_{n,0}, x_{n,1}, \cdots, x_{n,n}\}$ where $x_{n,i} = \left(\frac{2i}{n}\right)^3$. We have

$$\Delta_{n,i}x = x_{n,i} - x_{n,i-1} = \left(\tfrac{2i}{n}\right)^3 - \left(\tfrac{2(i-1)}{n}\right)^3 = \tfrac{8}{n^3}\left(i^3 - (i-1)^3\right) = \tfrac{8}{n^3}(3i^2 - 3i + 1)\,.$$

Note that $3i^2 - 3i + 1$ is increasing for $i \ge 1$ $\left(\text{since } g(x) = 3x^2 - 3x + 1 \text{ is increasing for } x \ge -\frac{1}{2}\right)$ and so we have $|X_n| = \Delta_{n,n}x = \frac{8}{n^3}(3n^2 - 3n + 1) \to 0$ as $n \to \infty$. Thus

$$\begin{aligned}
\int_0^\infty \sqrt[3]{x}\,dx &= \lim_{n\to\infty} \sum_{i=1}^n f(x_{n,i})\Delta_{n,i}x \\
&= \lim_{n\to\infty} \sum_{i=1}^n \left(\tfrac{2i}{n}\right)\left(\tfrac{8}{n^3}\right)(3i^2 - 3i + 1) \\
&= \lim_{n\to\infty} \left(\tfrac{48}{n^4}\sum_{i=1}^n i^3 + \tfrac{48}{n^4}\sum_{i=1}^n i^2 + \tfrac{16}{n^4}\sum_{i=1}^n i\right) \\
&= \lim_{n\to\infty} \left(\tfrac{48}{n^4}\,\tfrac{n^2(n+1)^2}{4} - \tfrac{48}{n^4}\,\tfrac{n(n+1)(2n+1)}{6} + \tfrac{16}{n^4}\,\tfrac{n(n+1)}{2}\right) \\
&= \tfrac{48}{4} - 0 + 0 \\
&= 12\,.
\end{aligned}$$

2: (a) Let f be increasing on $[a,b]$. Show that f is integrable on $[a,b]$.

Solution: Suppose that f is increasing $\big($and hence bounded, below by $f(a)$ and above by $f(b)\big)$ on $[a,b]$. Notice that since f is increasing we have $M_i = f(x_i)$ and $m_i = f(x_{i-1})$, where $M_i = \sup\left\{f(t)\big|t\in[x_{i-1},x_i]\right\}$ and $m_i = \inf\left\{f(t)\big|t\in[x_{i-1},x_i]\right\}$, and so $\sum_{i=1}^{n}(M_i - m_i) = \sum_{i=1}^{n}\left(f(x_i) - f(x_{i-1})\right) = f(x_n) - f(x_0) = f(b) - f(a)$. Now let $\epsilon > 0$. Choose a partition $X = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ with $|X| < \frac{\epsilon}{f(b)-f(a)}$. Then

$$\begin{aligned} U(f,X) - L(f,X) &= \sum_{i=1}^{n} M_i\Delta_i x - \sum_{i=1}^{n} m_i\Delta_i x \\ &= \sum_{i=1}^{n}(M_i - m_i)\Delta_i x \\ &\le \sum_{i=1}^{n}(M_i - m_i)|X| \\ &= \big(f(b) - f(a)\big)|X| \\ &< \big(f(b) - f(a)\big)\frac{\epsilon}{f(b) - f(a)} \\ &= \epsilon\,. \end{aligned}$$

Thus f is integrable on $[a,b]$.

(b) Define $f:[0,1]\to\mathbf{R}$ as follows. Let $f(0) = f(1) = 0$. For $x\in(0,1)$ with $x\notin\mathbf{Q}$, let $f(x) = 0$. For $x\in(0,1)$ with $x\in\mathbf{Q}$, write $x = \frac{a}{b}$ where $0 < a, b\in\mathbf{Z}$ with $\gcd(a,b) = 1$, and then let $f(x) = \frac{1}{b}$. Show that f is integrable in $[0,1]$.

Solution: Let $\epsilon > 0$ be arbitrary. Choose an integer $N > 0$ so that $\frac{1}{N} < \frac{\epsilon}{2}$. Note that there are only finitely many points $x\in[0,1]$ such that $f(x) > \frac{1}{N}$ (indeed the only such points are the points $x = \frac{a}{b}$ with $0 < a < b\in\mathbf{Z}$ with $b < N$). Say these points are $p_1, p_2, \cdots, p_{k-1}$ where

$$0 = p_0 < p_1 < p_2 < \cdots < p_{k-1} < p_k = 1\,.$$

Choose $\delta > 0$ so that $\delta < \frac{\epsilon}{2k}$ and so that $\delta < \frac{p_i - p_{i-1}}{2}$ for all $1 = 1, 2, \cdots, k$. Let X be the partition

$$X = \left\{0, p_1 - \delta, p_1 + \delta, p_2 - \delta, p_2 + \delta, \cdots, p_{k-1} - \delta, p_{k-1} + \delta, 1\right\}$$

Note that $L(f,X) = 0$ and since $f(x) \le \frac{1}{N}$ for all $x \ne p_i$, and $f(p_i) \le \frac{1}{2}$ for all $i = 1, 2, \cdots, k-1$, we have

$$\begin{aligned} U(f,X) &\le \tfrac{1}{N}(p_1 - \delta) + f(p_1)\cdot 2\delta + \tfrac{1}{N}(p_2 - p_1 - 2\delta) + f(p_2)\cdot 2\delta + \cdots + f(p_{k-1})\cdot 2\delta + \tfrac{1}{N}(1 - p_{k-1} - \delta) \\ &= \tfrac{1}{N}(1 - 2(k-1)\delta\,) + \big(f(p_1) + f(p_2) + \cdots + f(p_{k-1})\big)\cdot 2\delta \\ &< \tfrac{1}{N} + \tfrac{k-1}{2}\cdot 2\delta < \tfrac{1}{N} + k\,\delta < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,. \end{aligned}$$

3: (a) Show that if f is integrable on $[a,b]$ then f^2 is integrable on $[a,b]$.

Solution: Suppose that f is integrable on $[a,b]$. Then we know that $|f|$ is also integrable on $[a,b]$ (by the Estimation Theorem). Let M be an upper bound for $|f|$. Let $\epsilon > 0$ be arbitrary. Choose a partition X of $[a,b]$ so that $U(|f|,X) - L(|f|,X) < \frac{\epsilon}{2M}$. Note that $M_i(f^2) = M_i(|f|)^2$ and $m_i(f^2) = m_i(|f|)^2$ so we have

$$\begin{aligned} M_i(f^2) - m_i(f^2) &= M_i(|f|)^2 - m_i(|f|)^2 \\ &= \big(M_i(|f|) - m_i(|f|)\big)\big(M_i(|f|) + m_i(|f|)\big) . \\ &\leq \big(M_i(|f|) - m_i(|f|)\big) \cdot 2M \end{aligned}$$

Thus

$$\begin{aligned} U(f^2,X) - L(f^2,X) &= \sum_{i-1}^{n} \big(M_i(f^2) - m_i(f^2)\big)\Delta_i x \\ &\leq \sum_{i=1}^{n} \big(M_i(|f|) - m_i(|f|)\big) \cdot 2M \cdot \Delta_i x \\ &= 2M\big(U(|f|,X) - L(|f|,X)\big) < \epsilon\,. \end{aligned}$$

(b) Show that if f is integrable and non-negative on $[a,b]$, then $\sqrt{f}$ is integrable on $[a,b]$.

Solution: Suppose that f is integrable and non-negative on $[a,b]$. When $X = \{x_0, x_1, \cdots, x_n\}$ is a partition of $[a,b]$, let us write $M_i(\sqrt{f}) = \sup\big\{\sqrt{f(t)}\big| t \in [x_{i-1}, x_i]\big\}$ and $M_i(f) = \sup\big\{f(t)\big| t \in [x_{i-1}, x_i]\big\}$, and similarly for $m_i(\sqrt{f})$ and $m_i(f)$. Note that $M_i(f) = M_i(\sqrt{f})^2$ and $m_i(f) = m_i(\sqrt{f})^2$, and so we have

$$M_i(f) - m_i(f) = \big(M_i(\sqrt{f}) - m_i(\sqrt{f})\big)\big(M_i(\sqrt{f}) + m_i(\sqrt{f})\big)\,.$$

For any constant $c > 0$, when $M_i(\sqrt{f}) < c$ we have $M_i(\sqrt{f}) - m_i(\sqrt{f}) < c$, and when $M_i(\sqrt{f}) > c$ we have $M_i(\sqrt{f}) + m_i(\sqrt{f}) > c$ so that $M_i(f) - m_i(f) \geq \big(M_i(\sqrt{f}) - m_i(\sqrt{f})\big)\,c$, that is $M_i(\sqrt{f}) - m_i(\sqrt{f}) \leq \frac{1}{c}\big(M_i(f) - m_i(f)\big)$. Thus for any partition X and any constant $c > 0$ we have

$$\sum_{i \text{ such that } M_i(\sqrt{f}) < c} \big(M_i(\sqrt{f}) - m_i(\sqrt{f})\big)\,\Delta_i x \;\leq\; \sum_{i=1}^{n} c\,\Delta_i x = c\,(b-a)\ , \text{ and}$$

$$\sum_{i \text{ such that } M_i(\sqrt{f}) \geq c} \big(M_i(\sqrt{f}) - m_i(\sqrt{f})\big)\,\Delta_i x \;\leq\; \sum_{i=1}^{n} \tfrac{1}{c}\,\big(M_i(f) - m_i(f)\big)\,\Delta_i x = \tfrac{1}{c}\,\big(U(f,X) - L(f,X)\big)\,.$$

Now, let $\epsilon > 0$. Set $c = \dfrac{\epsilon}{2(b-a)}$ and choose a partition X of $[a,b]$ such that $U(f,X) - L(f,X) < \dfrac{\epsilon^2}{4(b-a)}$. Then

$$\begin{aligned} U(\sqrt{f},X) - L(\sqrt{f},X) &= \sum_{i=1}^{n} \big(M_i(\sqrt{f}) - m_i(\sqrt{f})\big)\,\Delta_i x \\ &= \sum_{i \text{ with } M_i(\sqrt{f}) < c} \big(M_i(\sqrt{f}) - m_i(\sqrt{f})\big)\,\Delta_i x \;+ \sum_{i \text{ with } M_i(\sqrt{f}) \geq c} \big(M_i(\sqrt{f}) - m_i(\sqrt{f})\big)\,\Delta_i x \\ &\leq\; c\,(b-a) + \tfrac{1}{c}\,\big(U(f,X) - L(f,X)\big) \\ &<\; \frac{\epsilon}{2(b-a)}\,(b-a) + \frac{2(b-a)}{\epsilon}\,\frac{\epsilon^2}{4(b-a)} = \epsilon\,. \end{aligned}$$

Thus $\sqrt{f}$ is integrable on $[a,b]$.