AMATH/PMATH 331 Real Analysis, Problems for Chapter 2

1: (a) Let $0 \leq a < b$. Let $f(x) = x^2$. From the definition of integrability, show that f is integrable on $[a, b]$ with $\int_a^b f = \frac{1}{3}(b^3 - a^3)$.

(b) Find $\int_0^8 \sqrt[3]{x}\, dx$ by evaluating the limit of a sequence of Riemann sums using the right endpoints of suitable partitions.

2: (a) Let f be increasing on $[a, b]$. Show that f is integrable on $[a, b]$.

(b) Define $f : [0, 1] \to \mathbf{R}$ as follows. Let $f(0) = f(1) = 0$. For $x \in (0, 1)$ with $x \notin \mathbf{Q}$, let $f(x) = 0$. For $x \in (0, 1)$ with $x \in \mathbf{Q}$, write $x = \frac{a}{b}$ where $0 < a, b \in \mathbf{Z}$ with $\gcd(a, b) = 1$, and then let $f(x) = \frac{1}{b}$. Show that f is integrable in $[0, 1]$.

3: (a) Show that if f is integrable on $[a, b]$ then f^2 is integrable on $[a, b]$.

(b) Show that if f is integrable and non-negative on $[a, b]$, then $\sqrt{f}$ is integrable on $[a, b]$.