AMATH/PMATH 331 Real Analysis, Solutions to the Problems for Chapter 1

1: Let $\mathbf{F} = \mathbf{Q}$ or $\mathbf{R}$. For $a, b \in \mathbf{F}$ with $a \leq b$ we write

$$(a,b) = \left\{x \in \mathbf{F} \middle| a < x < b\right\} \ , \ [a,b] = \left\{x \in \mathbf{F} \middle| a \leq x \leq b\right\},$$
$$(a,b] = \left\{x \in \mathbf{F} \middle| a < x \leq b\right\} \ , \ [a,b) = \left\{x \in \mathbf{F} \middle| a \leq x < b\right\}.$$

A **bounded interval** in $\mathbf{F}$ is any set of one of the above forms. For a subset $A \subseteq \mathbf{F}$, we say that A has the **intermediate value property** when for every $a, b, x \in \mathbf{F}$ with $a < x < b$, if $a \in A$ and $b \in A$ then $x \in A$.

(a) Find a bounded set $A \subseteq \mathbf{Q}$ which has the intermediate value property but which is not a bounded interval in $\mathbf{Q}$.

Solution: Let $A = \left\{x \in \mathbf{Q} \middle| 0 \leq x < \sqrt{2}\right\}$. Note that A is nonempty (since $0 \in A$) and bounded (A is bounded below in $\mathbf{Q}$ by 0 and A is bounded above in $\mathbf{Q}$ by 2). Also note that A has the intermediate value property because for $a, b, x \in \mathbf{Q}$ with $a < b < x$, if $a \in A$ and $b \in A$ then we have $0 \leq a < x < b < \sqrt{2}$ and so $x \in A$. We claim that A is not equal to any bounded interval in $\mathbf{Q}$. First note that since $A \neq \emptyset$, A is not equal to an interval of the form (a,a), $(a,a]$ or $[a,a)$, where $a \in \mathbf{Q}$, because these intervals are all empty. Next note that since A contains at least two points, A is not equal to an interval of the form $[a,a] = \{a\}$. Finally note that when I is equal to any one of the intervals (a,b), $(a,b]$, $[a,b)$ or $[a,b]$, where $a, b \in \mathbf{Q}$ with $a < b$, we have $\sup I = b \in \mathbf{Q}$, but $\sup A = \sqrt{2} \notin \mathbf{Q}$, and so $A \neq I$.

(b) Show that for every bounded set $A \subseteq \mathbf{R}$, if A has the intermediate value property then A is a bounded interval in $\mathbf{R}$.

Solution: Let A be a bounded set in $\mathbf{R}$, and suppose that A has the intermediate value property. If $A = \emptyset$ then A is equal to an interval of the form (a,a), which is empty. Suppose that A is not empty. Since A is nonempty and bounded in $\mathbf{R}$, it has an infimum and a supreme in $\mathbf{R}$. Let $a = \inf A$ and let $b = \sup A$. Note that $a \leq b$ since for any element $x \in A$, since $a = \inf A$ and $b = \sup A$ we have $a \leq x \leq b$. We claim that A is equal to one of the intervals (a,b), $(a,b]$, $[a,b)$ or $[a,b]$, depending on whether a or b or both or neither lie in A. We shall suppose that $a \in A$ and $b \notin A$ and prove that $A = [a,b)$ (the other three cases are similar). Let $x \in A$. Since $a = \inf A$ and $b = \sup A$ we have $a \leq x \leq b$. Since $x \in A$ but $b \notin A$ we have $x \neq b$, and so $a \leq x < b$. Since $a \in A \subseteq \mathbf{R}$ with $a \leq x < b$, we have $a \in [a,b)$. This shows that $A \subseteq [a,b)$. Conversely, suppose that $x \in [a,b)$, that is $x \in \mathbf{R}$ with $a \leq x < b$. If $x = a$ then $x \in A$ (since $a \in A$). If $x \neq a$ then we have $a < x < b$ and so $x \in A$ because A has the intermediate value property. In either case, $x \in A$. This shows that $[a,b) \subseteq A$.

2: (a) Let $x_k = \frac{2k+1}{k-1}$ for $k \geq 2$. Use the definition of the limit to show that $\lim\limits_{k\to\infty} x_k = 2$.

Solution: For $k \geq 2$ and $\epsilon > 0$, we have

$$|x_k - 2| = \left|\tfrac{2k+1}{k-1} - 2\right| = \left|\tfrac{2k+1-2k+2}{k-1}\right| = \tfrac{3}{k-1}$$

and

$$\tfrac{3}{k-1} < \epsilon \iff k - 1 > \tfrac{3}{\epsilon} \iff k > \tfrac{3}{\epsilon} + 1.$$

Let $\epsilon > 0$. Choose $m \in \mathbf{Z}$ with $m > \frac{3}{\epsilon} + 1$. For $k \in \mathbf{Z}_{\geq 2}$ with $k \geq m$ we have $k \geq m > \frac{3}{\epsilon} + 1$ and hence, as shown above, $|x_k - 2| = \frac{3}{k-1} < \epsilon$.

(b) Let $x_k = \frac{k}{\sqrt{k+3}}$ for $k \geq 0$. Use the definition of the limit to show that $\lim\limits_{n\to\infty} x_k = \infty$.

Solution: First note that for $k \geq 1$ we have $k + 3 \leq k + 3k = 4k$ and so

$$x_k = \tfrac{k}{\sqrt{k+3}} \geq \tfrac{k}{\sqrt{4k}} = \tfrac{\sqrt{k}}{2}.$$

Let $r \in \mathbf{R}$. Choose $m \in \mathbf{Z}$ with $m > 4r^2$. Then for $k \geq m$ we have $k > 4r^2$ and so

$$x_k \geq \tfrac{\sqrt{k}}{2} > \tfrac{\sqrt{4r^2}}{2} = \tfrac{2|r|}{2} = |r| \geq r.$$

(c) Let $x_k = \sin(k)$ for $k \geq 0$. Use the definition of the limit to show that $(x_k)_{k\geq 0}$ diverges.

Solution: Recall that $\sin(x + 2\pi t) = \sin x$ for all $x \in \mathbf{R}$ and all $t \in \mathbf{Z}$, and that $\sin x \geq \frac{1}{2}$ for all $x \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ and $\sin x \leq -\frac{1}{2}$ for all $x \in \left[\frac{7\pi}{6}, \frac{11\pi}{6}\right]$. Suppose, for a contradiction, that (x_k) converges. Let $a = \lim\limits_{k\to\infty} x_k$. Choose $m \in \mathbf{Z}$ so that $k \geq m \Longrightarrow |x_k - a| \leq \frac{1}{3}$. Choose $t \in \mathbf{N}$ with $t \geq \frac{m}{2\pi}$ so that $2\pi t \geq m$. Choose $k \in \mathbf{Z}$ with $k \in \left[2\pi t + \frac{\pi}{6}, 2\pi t + \frac{5\pi}{6}\right]$ (we can do this since the size of the interval is $\frac{5\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{3} > 1$ so, for example, we could choose $k = \left\lfloor 2\pi t + \frac{5\pi}{6} \right\rfloor$). Since $k - 2\pi t \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ we have $x_k = \sin k \geq \frac{1}{2}$, and since $k \geq 2\pi t \geq m$ we have $|x_k - a| \leq \frac{1}{3}$, and so we have $a \geq x_k - \frac{1}{3} \geq \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. Now choose $l \in \mathbf{Z}$ with $l \in \left[2\pi t + \frac{7\pi}{6}, 2\pi t + \frac{11\pi}{6}\right]$. Then, as above, we have $x_l = \sin l \leq -\frac{1}{2}$ and $|x_l - a| \leq \frac{1}{3}$ and so $a \leq x_l + \frac{1}{3} \leq -\frac{1}{2} + \frac{1}{3} = -\frac{1}{6}$. This is not possible since we cannot have $a \leq -\frac{1}{6}$ and $a \geq \frac{1}{6}$.

(d) Let $x_1 = \frac{7}{2}$ and for $k \geq 1$ let $x_{k+1} = \dfrac{6}{5 - x_k}$. Find $\lim\limits_{k\to\infty} x_k$ if it exists.

Solution: Suppose for now that (x_k) does converge, and say $\lim\limits_{n\to\infty} x_k = a$. Then we also have $\lim\limits_{k\to\infty} x_{k+1} = a$ and so taking the limit on both sides of the recursion formula $x_{k+1} = \frac{6}{5-x_k}$ gives

$$a = \frac{6}{5-a} \Longrightarrow 5a - a^2 = 6 \Longrightarrow a^2 - 5a + 6 = 0 \Longrightarrow (a-2)(a-3) = 0,$$

and so we must have $a = 2$ or $a = 3$.

We claim that $x_n < x_{n+1} < 2$ for all $n \geq 4$. We have $x_1 = \frac{7}{2}$, $x_2 = 4$, $x_3 = 6$, $x_4 = -6$ and $x_5 = \frac{6}{11}$, so the claim is true when $n = 4$. Suppose the claim is true when $n = k$. Then we have

$$x_k < x_{k+1} < 2 \Longrightarrow -x_k > -x_{k+1} > -2 \Longrightarrow 5 - x_k > 5 - x_{k+1} > 3 \Longrightarrow \tfrac{1}{5-x_k} < \tfrac{1}{5-x_{k+1}} < \tfrac{1}{3}$$
$$\Longrightarrow \tfrac{6}{5-x_k} < \tfrac{6}{5-x_{k+1}} < 2 \Longrightarrow x_{k+1} < x_{k+2} < 2,$$

so the claim is true when $n = k+1$. By induction, the claim is true for all $n \geq 4$. Thus $(x_n)_{n\geq 4}$ is increasing and is bounded above by 2, so (x_n) converges and $\lim\limits_{n\to\infty} x_n \leq 2$ by the Monotone Convergence Theorem. We showed above that the limit must be 2 or 3, and so we must have $\lim\limits_{n\to\infty} x_n = 2$.

3: (a) Find a divergent sequence $(x_k)_{k\geq 0}$ in $\mathbf{R}$ with $\left|x_k - x_{k-1}\right| \leq \frac{1}{k}$ for all $k \geq 1$.

Solution: Let $x_0 = 0$ and for $k \geq 1$, let $x_k = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$. Note that $|x_k - x_{k-1}| = x_k - x_{k-1} = \frac{1}{k}$ for all $k \geq 1$. Consider the subsequence $(x_{2^k})_{k\geq 0} = (x_1, x_2, x_4, x_8, \cdots)$. We have $x_{2^0} = x_1 = 1$. Let $k \geq 0$ and suppose, inductively, that $x_{2^k} \geq 1 + \frac{k}{2}$. Then

$$\begin{aligned}
x_{2^{k+1}} &= \left(\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{2^k}\right) + \left(\tfrac{1}{2^k+1} + \tfrac{1}{2^k+2} + \cdots + \tfrac{1}{2^{k+1}}\right) \\
&= x_{2^k} + \left(\tfrac{1}{2^k+1} + \tfrac{1}{2^k+2} + \cdots + \tfrac{1}{2^{k+1}}\right) \\
&\geq x_{2^k} + \left(\tfrac{1}{2^{k+1}} + \tfrac{1}{2^{k+1}} + \cdots + \tfrac{1}{2^{k+1}}\right) \\
&= x_{2^k} + 2^k \cdot \tfrac{1}{2^{k+1}} = x_{2^k} + \tfrac{1}{2} \geq 1 + \tfrac{k}{2} + \tfrac{1}{2} = 1 + \tfrac{k+1}{2}.
\end{aligned}$$

By induction, we have $x_{2^k} \geq 1 + \frac{k}{2}$ for all $k \geq 0$. Since $x_{2^k} \geq 1 + \frac{k}{2}$, it follows that (x_k) is not bounded (indeed, given $r \in \mathbf{R}$ we can choose $k \geq 0$ so that $1 + \frac{k}{2} \geq r$ and then we have $x_{2^k} \geq 1 + \frac{k}{2} \geq r$). Since (x_k) is increasing and unbounded, we have $x_k \to \infty$, by the Monotone Convergence Theorem.

(b) Let $(x_k)_{k\geq 0}$ be a sequence in $\mathbf{R}$ with $\left|x_k - x_{k-1}\right| \leq \frac{1}{k^2}$ for all $k \geq 1$. Show that (x_k) converges in $\mathbf{R}$.

Solution: Notice that for all $k \geq 2$ we have $\frac{1}{k^2} \leq \frac{1}{(k-1)\,k} = \frac{1}{k-1} - \frac{1}{k}$. It follows that for $1 \leq k < l$ we have

$$\begin{aligned}
|x_k - x_l| &= \left|x_k - x_{k+1} + x_{k+1} - x_{k+2} + x_{k+2} - x_{k+3} + \cdots - x_{l-1} + x_{l-1} - x_l\right| \\
&\leq |x_k - x_{k+1}| + |x_{k+1} - x_{k+2}| + |x_{k+2} - x_{k+3}| + \cdots + |x_{l-1} - x_l| \\
&\leq \tfrac{1}{(k+1)^2} + \tfrac{1}{(k+2)^2} + \tfrac{1}{(k+3)^2} + \cdots + \tfrac{1}{(l-1)^2} + \tfrac{1}{l^2} \\
&\leq \tfrac{1}{k(k+1)} + \tfrac{1}{(k+1)(k+2)} + \tfrac{1}{(k+2)(k+3)} + \cdots + \tfrac{1}{(l-2)(l-1)} + \tfrac{1}{(l-1)\,l} \\
&= \tfrac{1}{k} - \tfrac{1}{k+1} + \tfrac{1}{k+1} - \tfrac{1}{k+2} + \tfrac{1}{k+2} - \tfrac{1}{k+3} + \cdots - \tfrac{1}{l-1} + \tfrac{1}{l-1} - \tfrac{1}{l} \\
&= \tfrac{1}{k} - \tfrac{1}{l} \leq \tfrac{1}{k}.
\end{aligned}$$

Let $\epsilon > 0$. Choose $m \in \mathbf{Z}$ with $m \geq \frac{1}{\epsilon}$. For $k, l \geq m$ say with $k \leq l$, if $k = l$ then $|x_k - x_l| = 0$ and if $k < l$ then, as shown above, $|x_k - x_l| \leq \frac{1}{k} \leq \frac{1}{m} \leq \epsilon$. Thus (x_k) is a Cauchy sequence, and so it converges by the Cauchy Criterion.

4: (a) Show that every sequence (x_k) in $\mathbf{R}$ has a monotonic subsequence. Hint: consider indices k with the property that $x_k > x_j$ for all $j > k$.

Solution: For an index k, let us say that k is a **peak** index of (x_k) when it has the property that $x_k > x_j$ for all $j > k$. Either (x_k) has infinitely many peak indices, or it does not. If (x_k) has infinitely many peak indices, then we can choose peak indices $k_1 < k_2 < k_3 < \cdots$ and then, by the definition of a peak index, $x_{k_1} > x_{k_2} > x_{k_3} > \cdots$. Suppose that (x_k) has only finitely many peak indices. Choose an index k_1 which is greater than every peak index. Since k_1 is not a peak index, we can choose $k_2 > k_1$ so that $x_{k_2} \geq x_{k_1}$. Since k_2 is greater than k_1 which is greater than every peak index, k_2 is not a peak index and so we can choose $k_3 > k_2$ so that $x_{k_3} \geq x_{k_2}$. We continue this process to obtain indices $k_1 < k_2 < k_3 < \cdots$ with $x_{k_1} \leq x_{k_2} \leq x_{k_3} \leq \cdots$.

We remark that it is also possible to ignore the hint and prove this result by using the Bolzano-Weierstrass Theorem. To do this, consider several cases. When (x_k) is not bounded above, construct an increasing subsequence of (x_k). When (x_k) is not bounded below, construct a decreasing subsequence. When (x_k) is bounded, invoke the Bolzano-Weierstrass Theorem to choose a convergent subsequence (x_{k_l}) and say $u_l = x_{k_l} \to a$. Then consider the following three cases. Either there exist infinitely many indices l with $u_l = a$ (in this case, construct a constant subsequence of (u_l)) or there exist infinitely many indices l with $u_l > a$ (in this case, construct a decreasing subsequence of (u_l)) or there exist infinitely many indices l with $u_l < a$ (in this case, construct an increasing subsequence of (u_l)).

We also remark that the fact that every sequence in $\mathbf{R}$ has a monotonic subsequence, together with the Monotone Convergence Theorem, immediately imply the Bolzano-Weierstrass Theorem as a corollary. Thus the first solution to this problem supplies you with an alternate (and perhaps easier) proof of the Bolzano-Weirstrass Theorem than the proof we gave (which made use of the Nested Interval Property of $\mathbf{R}$).

(b) Let $x_k = \frac{k}{\sqrt{2}} - \lfloor \frac{k}{\sqrt{2}} \rfloor$ for $k \geq 0$. Show that (x_k) has a monotonic subsequence (x_{k_j}) with $x_{k_j} \to 0$ as $j \to \infty$.

Solution: By the Binomial Theorem, we have

$$\begin{aligned}(1+\sqrt{2})^n &= 1 + \binom{n}{1}(\sqrt{2}) + \binom{n}{2}(\sqrt{2})^2 + \binom{n}{3}(\sqrt{2})^3 + \binom{n}{4}(\sqrt{2})^4 + \cdots \quad \text{and} \\ (1-\sqrt{2})^n &= 1 - \binom{n}{1}(\sqrt{2}) + \binom{n}{2}(\sqrt{2})^2 - \binom{n}{3}(\sqrt{2})^3 + \binom{n}{4}(\sqrt{2})^4 - \cdots\end{aligned}$$

hence

$$\begin{aligned}(1+\sqrt{2})^n + (1-\sqrt{2})^n &= 2\Big(1 + \binom{n}{2}\cdot 2 + \binom{n}{4}\cdot 2^2 + \binom{n}{6}\cdot 2^3 + \cdots\Big) \quad \text{and} \\ (1+\sqrt{2})^n - (1-\sqrt{2})^n &= 2\sqrt{2}\Big(\binom{n}{1} + \binom{n}{3}(2) + \binom{n}{5}(2)^2 + \binom{n}{7}2^3 + \cdots\Big),\end{aligned}$$

and so we see that $\frac{1}{2}\big((1+\sqrt{2})^n + (1-\sqrt{2})^n\big) \in \mathbf{Z}$ and $\frac{1}{\sqrt{2}}\big((1+\sqrt{2}) - (1-\sqrt{2})^n\big) \in \mathbf{Z}$ for all $n \in \mathbf{N}$. For each $n \in \mathbf{N}$, let

$$k_n = \tfrac{1}{\sqrt{2}}\big((1+\sqrt{2})^n - (1-\sqrt{2})^n\big)$$

and note that $k_n \in \mathbf{Z}$. Consider the case that $n \in \mathbf{N}$ is odd. Since $-1 < (1-\sqrt{2})^n < 0$ and

$$\tfrac{k_n}{\sqrt{2}} + (1-\sqrt{2})^n = \tfrac{1}{2}\big((1+\sqrt{2})^n - (1-\sqrt{2})^n\big) + (1-\sqrt{2})^n = \tfrac{1}{2}\big((1+\sqrt{2})^n + (1-\sqrt{2})^n\big) \in \mathbf{Z},$$

it follows that $\lfloor \frac{k_n}{\sqrt{2}} \rfloor = \frac{1}{2}\big((1+\sqrt{2})^n + (1-\sqrt{2})^n\big)$. Thus, when n is odd, we have

$$x_{k_n} = \frac{k_n}{\sqrt{2}} - \left\lfloor \frac{k_n}{\sqrt{2}} \right\rfloor = \tfrac{1}{2}\big((1+\sqrt{2})^n - (1-\sqrt{2})^n\big) - \tfrac{1}{2}\big((1+\sqrt{2})^n + (1-\sqrt{2})^n\big) = -(1-\sqrt{2})^n = (\sqrt{2}-1)^n.$$

Thus the subsequence $x_{k_1}, x_{k_3}, x_{k_5}$ of (x_k) is equal to the sequence $(\sqrt{2}-1), (\sqrt{2}-1)^3, (\sqrt{2}-1)^5, \cdots$ which is decreasing with limit 0.

We remark that one can also prove the result of Problem 4(b) by first proving the follow more general result. Define $f : \mathbf{R} \to [0, 1)$ by $f(x) = x - \lfloor x \rfloor$ ($f(x)$ is called the **fractional part** of x). Let $\alpha \in \mathbf{R}$. Define $x_k = f(\alpha k)$ for $k \geq 0$. If $\alpha \in \mathbf{Q}$ then the sequence (x_k) is periodic. If $\alpha \notin \mathbf{Q}$ then

$$\forall a \in [0,1] \;\; \forall \epsilon > 0 \;\; \forall m \in \mathbf{Z}^+ \;\; \exists k \geq m \;\; |x_k - a| \leq \epsilon\,.$$

We sketch a proof below. We leave it as an exercise to show that 4(b) follows as a corollary.

From the definition of the floor function and the fractional part function $f(x)$, verify that

$$f(x+y) = \begin{cases} f(x) + f(y) & \text{if } f(x) + f(y) < 1 \\ f(x) + f(y) - 1 & \text{if } f(x) + f(y) \geq 1 \end{cases}$$

and

$$f(x-y) = \begin{cases} f(x) - f(y) & \text{if } f(x) \geq f(y) \\ f(x) - f(y) + 1 & \text{if } f(x) < f(y). \end{cases}$$

Since $x_k = f(\alpha k)$, these formulas imply that

$$x_{k_1+k_2} = \begin{cases} x_{k_1} + x_{k_2} & \text{if } x_{k_1} + x_{k_2} < 1 \\ x_{k_1} + x_{k_2} - 1 & \text{if } x_{k_1} + x_{k_2} \geq 1. \end{cases}$$

and

$$x_{k_1-k_2} = \begin{cases} x_{k_2} - x_{k_1} & \text{if } x_{k_2} \geq x_{k_1} \\ x_{k_2} - x_{k_1} + 1 & \text{if } x_{k_2} < x_{k_1}. \end{cases}$$

We wish to prove that when $\alpha \notin \mathbf{Q}$,

$$\forall a \in [0,1] \;\; \forall \epsilon > 0 \;\; \forall m \in \mathbf{Z}^+ \;\; \exists k \geq m \;\; |x_k - a| \leq \epsilon.$$

Let $a \in [0, 1]$ and let $\epsilon > 0$. Choose $n \in \mathbf{Z}^+$ so that $\frac{1}{n} \geq \epsilon$, then divide the interval $[0, 1]$ into the n subintervals $I_j = \left[\frac{j-1}{n}, \frac{j}{n}\right]$, and note that each of these intervals is of size $\frac{j}{n} - \frac{j-1}{n} = \frac{1}{n}$. Since $a \in [0,1] = \bigcup_{j=1}^{n} I_j$, we can choose an index $j \in \{1, 2, \cdots, n\}$ such that $a \in I_j$. Since the interval I_j is of size $\frac{1}{n} \leq \epsilon$, it suffices to show that for all $m \in \mathbf{Z}^+$ we can find $k \geq m$ so that $x_k \in I_j$ (because when x_k and a both lie in the same interval I_j we must have $|x_k - a| \leq \frac{1}{n} \leq \epsilon$). It remains for us to show that

$$\forall m \in \mathbf{Z}^+ \;\; \exists k \geq m \;\; x_k \in I_j = \left[\tfrac{j-1}{n}, \tfrac{j}{n}\right].$$

Let $m \in \mathbf{Z}^+$. Choose an index $j_0 \in \{1, 2, \cdots, n\}$ so that for infinitely many indices k we have $x_k \in I_{j_0}$. Choose two indices $k_1, k_2 \in \mathbf{Z}^+$ with $k_2 \geq k_1 + m$ such that $x_{k_1}, x_{k_2} \in I_{j_0}$, and let $l = k_2 - k_1 \geq m$. From our formula for $x_{k_1-k_2}$, we have

$$x_l = x_{k_1-k_2} = \begin{cases} x_{k_2} - x_{k_1} \in \left[0, \frac{1}{n}\right] & \text{if } x_{k_2} \geq x_{k_1} \\ x_{k_2} - x_{k_1} + 1 \in \left[1 - \frac{1}{n}, 1\right] & \text{if } x_{k_2} < x_{k_1}. \end{cases}$$

We have found an index $l \geq m$ such that $x_l \in \left[0, \frac{1}{n}\right] \cup \left[1 - \frac{1}{n}, 1\right]$. We shall show that there is a multiple $k = tl$, where $t \in \mathbf{Z}^+$, such that $x_k \in I_j$ where I_j was the interval that we chose earlier with $a \in I_j$. Since $\alpha \notin \mathbf{Q}$, we have $k\alpha \notin \mathbf{Q}$ for all $k \in \mathbf{Z}^+$ and hence $x_k = f(\alpha k) = \alpha k - \lfloor \alpha k \rfloor \notin \mathbf{Q}$. It follows that $x_l \in \left(0, \frac{1}{n}\right) \cup \left(1 - \frac{1}{n}, 1\right)$. Suppose first that $x_l \in \left(0, \frac{1}{n}\right)$. From our formula for $x_{k_1+k_2}$ we see that $x_{tl} = t\,x_l$ as long as $t\,x_l < 1$. Since $0 < x_k < \frac{1}{n}$, we can choose $t \in \mathbf{Z}^+$ so that $t\,x_l \in I_j$ (to be explicit, verify that if we choose $t = \left\lfloor \frac{j}{n\,x_l} \right\rfloor$ then we have $t\,x_l \in I_j$). Then we let $k = tl$ and we have found an index $k \geq m$ such that $x_k \in I_j$. The case that $x_l \in \left(1 - \frac{1}{n}, 1\right)$ is quite similar. If we write $x_l = 1 - \delta$ then we have $0 < \delta < \frac{1}{n}$. From the formula for $x_{k_1+k_2}$ we see that $x_{tl} = 1 - t\delta$ as long as $t\delta \leq 1$. Since $0 < \delta < \frac{1}{n}$, we can choose $t \in \mathbf{Z}^+$ so that $1 - t\delta \in I_j$. Then we let $k = tl$ so that $x_k \in I_j$. This completes the proof that for all $m \in \mathbf{Z}^+$ there exists $k \geq m$ such that $x_k \in I_j$, and the proof of our original claim that

$$\forall a \in [0,1] \;\; \forall \epsilon > 0 \;\; \forall m \in \mathbf{Z}^+ \;\; \exists k \geq m \;\; |x_k - a| \leq \epsilon\,.$$

5: (a) Show that there exist (at least) 3 distinct values of x such that $8x^3 = 6x + 1$.

Solution: Let $f(x) = 8x^3 - 6x - 1$. Notice that $f(x)$ is continuous and we have $f(x) = 0 \iff 8x^3 = 6x+1$. By the Intermediate Value Theorem, since $f(-1) = -3 < 0$ and $f\big(-\frac{1}{2}\big) = 1 > 0$, there is a number $x_1 \in \big(-1, -\frac{1}{2}\big)$ such that $f(x_1) = 0$. Similarly, since $f\big(-\frac{1}{2}\big) = 1 > 0$ and $f(0) = -1 < 0$, there is a number $x_2 \in \big(-\frac{1}{2}, 0\big)$ with $f(x_2) = 0$, and since $f(0) = -1 < 0$ and $f(1) = 1 > 0$, there is a number $x_3 \in (0, 1)$ with $f(x_3) = 0$. $\big($In fact, the exact values of x_1, x_2 and x_3 are $x_1 = -\cos(40°)$, $x_2 = -\sin(10°)$ and $x_3 = \cos(20°)\big)$.

(b) Let $f : [0, 2] \to \mathbf{R}$ be continuous with $f(0) = f(2)$. Show that $f(x) = f(x + 1)$ for some $x \in [0, 1]$.

Solution: Let $g(x) = f(x + 1) - f(x)$. Note that g is continuous and

$$g(1) = f(2) - f(1) = f(0) - f(1) = -\big(f(1) - f(0)\big) = -g(0).$$

By the Intermediate Value Theorem, there is a number $x \in [0, 1]$ with $g(x) = 0$ $\big($indeed if $g(0) \neq 0$ then one of the numbers $g(0)$ and $g(1)$ is positive and the other is negative so there is a number $x \in (0, 1)$ with $g(x) = 0\big)$. Then we have $0 = g(x) = f(x + 1) - f(x)$ and so $f(x) = f(x + 1)$.

(c) Let $f : \mathbf{R} \to \mathbf{R}$ be continuous. Suppose that $|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbf{R}$. Show that f is surjective.

Solution: First we note that f is injective since when $x \neq y$ we have $|f(x) - f(y)| \geq |x - y| > 0$ so that $f(x) \neq f(y)$. Consider the two intervals $I = [0, \infty)$ and $J = (-\infty, 0]$. We claim that the image $f(I)$ entirely contains one of the two intervals $[f(0), \infty)$ and $(-\infty, f(0)]$. Since the set $\mathbf{Z}^+$ is infinite and f is injective, either there exist infinitely many $k \in \mathbf{Z}^+$ such that $f(k) > f(0)$ or there exist infinitely many $k \in \mathbf{Z}^+$ such that $f(k) < f(0)$. Consider the case that there exist infinitely many $k \in \mathbf{Z}^+$ such that $f(k) > f(0)$. We claim that, in this case, we have $[f(0), \infty) \subseteq f(I)$. Choose $k_1 < k_2 < k_3 < \cdots$ such that $f(k_j) > f(0)$ for every index j. For every index j, since $f(k_j) > f(0)$ and $\big|f(k_j) - f(0)\big| \geq |k_j - 0| = k_j$, we have $f(k_j) > f(0) + k_j$. Let $y \in [f(0), \infty)$. Choose j with $k_j \geq y + f(0)$ so that we have $f(k_j) \geq f(0) + k_j \geq y$. Since f is continuous and $f(0) \leq y \leq f(k_j)$, it follows from the Intermediate Value Theorem that we can choose $x \in [0, k_j]$ such that $f(x) = y$. This proves our claim that $[f(0), \infty) \subseteq f(I)$. Similarly, in the case that there exist infinitely many $k \in \mathbf{Z}^+$ with $f(k) < f(0)$ we have $(-\infty, f(0)] \subseteq f(I)$. Thus one of the two intervals $K = [f(0), \infty)$ and $L = (-\infty, f(0)]$ is entirely contained in $f(I)$. A similar argument shows that one of the two intervals K and L is entirely contained in $f(J)$. Since f is injective, it is not possible that one of K and L can be contained in both of $f(I)$ and $f(J)$ (for example if we had $K \subseteq f(I) \cap f(L)$, then given $f(0) \neq y \in K$ we could choose $0 \neq x_1 \in I$ and $0 \neq x_2 \in J$ with $f(x_1) = y = f(x_2)$). Thus K is contained in one of the sets $f(I)$ and $f(J)$, and L is contained in the other. Thus we have $\mathbf{R} = K \cup L \subseteq f(I) \cup f(J) = f(I \cup J) = f(\mathbf{R})$, or in other words, f is surjective.

6: (a) Define $f, g : \mathbf{R} \to \mathbf{R}$ by $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$. Show that g is uniformly continuous but that f is not.

Solution: We claim that $f(x)$ is not uniformly continuous. Choose $\epsilon = 1$. Let $\delta > 0$ Choose $a = \frac{1}{\delta}$ and $x = \delta + \frac{1}{\delta}$. Then $|x - a| = \delta$ and we have

$$\big|f(x) - f(a)\big| = \left(\delta + \tfrac{1}{\delta}\right)^3 - \left(\tfrac{1}{\delta}\right)^3 = 3\delta + 3 \cdot \tfrac{1}{\delta} + \delta^3 \geq 3\left(\delta + \tfrac{1}{\delta}\right) \geq 3 > \epsilon$$

because when $\delta \geq 1$ we have $\delta + \frac{1}{\delta} \geq \delta \geq 1$ and when $0 < \delta \leq 1$ we have $\delta + \frac{1}{\delta} \geq \frac{1}{\delta} \geq 1$. Thus f is not uniformly continuous.

We claim that g is uniformly continuous. First we note that for $\delta > 0$ and for $a, x \in \mathbf{R}$, in the case that $|a| \leq 2\delta$, when $|x - a| \leq \delta$ we have $|x| \leq 3\delta$ and so

$$|f(x) - f(a)| \leq |f(x)| + |f(a)| \leq (2\delta)^{1/3} + (3\delta)^{1/3} = (2^{1/3} + 3^{1/3})\,\delta^{1/3} \leq 3\,\delta^{1/3}$$

and in the case that $|a| \geq 2\delta$, when $|x - a| \leq \delta$, the numbers a and x have the same sign and we have $|x| \geq \delta$ and so

$$\begin{aligned} |f(x) - f(a)| = |x^{1/3} - a^{1/3}| &= \left|\frac{x - a}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}}\right| = \frac{|x - a|}{|x|^{2/3} + |x|^{1/3}|a|^{1/3} + |a|^{2/3}} \\ &\leq \frac{\delta}{\delta^{2/3} + \delta^{1/3}(2\delta)^{1/3} + (2\delta)^{2/3}} = \frac{\delta^{1/3}}{1 + 2^{1/3} + 4^{1/3}} \leq \delta^{1/3} \leq 3\,\delta^{1/3}. \end{aligned}$$

Thus given $\epsilon > 0$ we can choose $\delta = \frac{1}{27}\epsilon^3$ so that $3\,\delta^{1/3} = \epsilon$ and then for all $a, x \in \mathbf{R}$ with $|x - a| \leq \delta$ we have $|f(x) - f(a)| \leq 3\delta^{1/3} = \epsilon$. Thus g is uniformly continuous.

(b) Find an example of a function $f : \mathbf{R} \to \mathbf{R}$ which is continuous and bounded but not uniformly continuous.

Solution: We wish to construct a function f whose graph oscillates more and more rapidly as x increases. Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = \cos(\pi x^2)$. Note that f is continuous (because it is elementary) and we have $f(\sqrt{n}) = \cos(\pi n) = (-1)^n$ for all $n \in \mathbf{Z}^+$. We claim that f is not uniformly continuous. Choose $\epsilon = 1$. Let $\delta > 0$. Since $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0$ as $n \to \infty$, we can choose $n \in \mathbf{Z}^+$ so that $\sqrt{n+1} - \sqrt{n} \leq \delta$. Then for $a = \sqrt{n}$ and $x = \sqrt{n+1}$ we have $|x - a| \leq \delta$ but $|f(x) - f(a)| = \big|(-1)^{n+1} - (-1)^n\big| = 2 > \epsilon$.

(c) Let $a, b \in \mathbf{R}$ with $a < b$, and let $f, g : [a, b] \to \mathbf{R}$. Suppose that f and g are both uniformly continuous and bounded. Show that fg is uniformly continuous.

Solution: Let $\epsilon > 0$. Since f and g are bounded, we can choose $m \geq 0$ so that $|f(x)| \leq m$ and $|g(x)| \leq m$ for all $x \in [a, b]$. Since f and g are uniformly continuous, we can choose $\delta > 0$ so that for all $x, y \in [a, b]$ with $|x - y| \leq \delta$ we have $|f(x) - f(y)| \leq \frac{\epsilon}{2m}$ and $|g(x) - g(y)| \leq \frac{\epsilon}{2m}$. Then for $|x - y| \leq \delta$ we have

$$\begin{aligned} \big|(fg)(x) - (fg)(y)\big| &= \big|f(x)g(x) - f(y)g(y)\big| \\ &= \big|f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)\big| \\ &\leq \big|f(x)g(x) - f(x)g(y)\big| + \big|f(x)g(y) - f(y)g(y)\big| \\ &= |f(x)|\,|g(x) - g(y)| + |g(y)|\,|f(x) - f(y)| \\ &\leq m \cdot \tfrac{\epsilon}{2m} + m \cdot \tfrac{\epsilon}{2m} = \epsilon. \end{aligned}$$

Thus fg is uniformly continuous, as required.

7: (a) Let $f : \mathbf{R} \to \mathbf{R}$ be differentiable with $f(0) = 3$. Suppose $f'(x) \leq 1$ for all $x > 0$. Prove that there is a number $a > 0$ such that $f(a) = 2a$.

Solution: Given $x > 0$, by the Mean Value Theorem we can choose $c \in (0, x)$ with$f'(c) = \frac{f(x)-f(0)}{x-0} = \frac{f(x)-3}{x}$, that is $f(x) = f'(c) \cdot x + 3$. Since $f'(c) \leq 1$ and $x > 0$ we have $f(x) = f'(c) \cdot x + 3 \leq x + 3$. This shows that $f(x) \leq x + 3$ for all $x \geq 0$. In particular, we have $f(3) \leq 6$.

Let $g(x) = f(x) - 2x$. Then g is differentiable in $\mathbf{R}$ with $g(0) = f(0) = 3$ and $g(3) = f(3) - 6 \leq 6 - 6 = 0$. Since $g(3) \leq 0 \leq g(0)$, by the Intermediate Value Theorem we can choose $a \in [0, 3]$ such that $g(a) = 0$. Then we have $0 = g(a) = f(a) - 2a$ and so $f(a) = 2a$, as required.

(b) Let $f : \mathbf{R} \to \mathbf{R}$ be twice differentiable with $f(0) = 0$ and $f(1) = 1$ and $f'(0) = f'(1) = 0$. Show that $|f''(x)| \geq 4$ for some $x \in [0, 1]$.

Solution: Suppose, for a contradiction, that $|f''(x)| < 4$ for all $x \in [0, 1]$. Let $g(x) = 2x^2 - f(x)$. Then g is twice differentiable in $\mathbf{R}$ with $g'(x) = 4x - f'(x)$ and $g''(x) = 4 - f''(x)$ for all $x \in \mathbf{R}$ and with $g(0) = 0$ and $g'(0) = 0$. Since $f''(x) \leq |f''(x)| < 4$ for all $x \in [0, 1]$, we have $g''(x) = 4 - f''(x) > 0$ for all $x \in [0, 1]$, and so $g'(x)$ is strictly increasing on $[0, 1]$. Since $g'(0) = 0$ and $g'(x)$ is strictly increasing, we have $g'(x) > 0$ for all $x \in (0, 1]$, and so $g(x)$ is strictly increasing on $[0, 1]$. In particular we have $0 = g(0) < g\left(\frac{1}{2}\right) = \frac{1}{2} - f\left(\frac{1}{2}\right)$ and so we have $f\left(\frac{1}{2}\right) < \frac{1}{2}$.

Let $h(x) = f(x) - \left(1 - 2(x-1)^2\right) = f(x) + 2x^2 - 4x + 1$. Then h is twice differentiable in $\mathbf{R}$ with $h'(x) = f'(x) + 4x - 4$ and $h''(x) = f''(x) + 4$ and with $h(1) = 0$ and $h'(1) = 0$. Since $f''(x) \geq -|f''(x)| > -4$ for all $x \in [0, 1]$, we have $h''(x) = f''(x) + 4 > 0$ for all $x \in [0, 1]$, and so $h'(x)$ is strictly increasing on $[0, 1]$. Since $h'(1) = 0$ and h' is increasing on $[0, 1]$, we have $h'(x) < 0$ for all $x \in [0, 1)$, and so h is strictly decreasing on $[0, 1]$. In particular we have $0 = h(1) < h\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) - \frac{1}{2}$ and so we have $f\left(\frac{1}{2}\right) > \frac{1}{2}$. This gives the desired contradiction.

(c) Prove that ${\sqrt{x}}^{\sqrt{x+1}} > {\sqrt{x+1}}^{\sqrt{x}}$ for all $x > e^2$.

Solution: Note first that for $x > 0$ we have

$$\begin{aligned} {\sqrt{x}}^{\sqrt{x+1}} > {\sqrt{x+1}}^{\sqrt{x}} &\iff \ln\left({\sqrt{x}}^{\sqrt{x+1}}\right) > \ln\left({\sqrt{x+1}}^{\sqrt{x}}\right) \\ &\iff \sqrt{x+1}\,\ln\sqrt{x} > \sqrt{x}\,\ln\sqrt{x+1} \\ &\iff \frac{\ln\sqrt{x}}{\sqrt{x}} > \frac{\ln\sqrt{x+1}}{\sqrt{x+1}}. \end{aligned}$$

Let $f(x) = \dfrac{\ln\sqrt{x}}{\sqrt{x}} = \dfrac{\frac{1}{2}\ln x}{\sqrt{x}}$. Then $f'(x) = \dfrac{\frac{1}{2x} \cdot \sqrt{x} - \frac{1}{2}\ln x \cdot \frac{1}{2\sqrt{x}}}{x} = \dfrac{2 - \ln x}{4x\sqrt{x}}$ and so

$$f'(x) < 0 \iff 2 - \ln x < 0 \iff \ln x > 2 \iff x > e^2.$$

Thus the function $f(x)$ is strictly decreasing for $x \geq e^2$ and so, in particular, when $x > e^2$ we have $f(x) > f(x+1)$, that is $\dfrac{\ln\sqrt{x}}{\sqrt{x}} > \dfrac{\ln\sqrt{x+1}}{\sqrt{x+1}}$, as required.

8: (a) Let $f:\mathbf{R}\to\mathbf{R}$ be differentiable with $\lim_{x\to\infty} f'(x)=b$. Show that $\lim_{x\to\infty}\big(f(x+1)-f(x)\big)=b$.

Solution: Let (x_k) be a sequence in $\mathbf{R}$ with $x_k\to\infty$. For each index k, by the Mean Value Theorem we can choose $c_k\in[x_k,x_k+1]$ so that $f'(c_k)=f(x_k+1)-f(x_k)$. Since $x_k\to\infty$ and $c_k\geq x_k$ for all k, we have $c_k\to\infty$ by the Comparison Theorem. Since $\lim_{x\to\infty} f'(x)=b$ and $c_k\to\infty$, it follows from the Sequential Characterization of Limits that $f'(c_k)\to b$, and so we have $f(x_k+1)-f(x_k)=f'(c_k)\to b$. We have shown that for every sequence (x_k) in $\mathbf{R}$ with $x_k\to\infty$ we have $f(x_k+1)-f(x_k)\to b$. It follows from another appeal to the Sequential Characterization of Limits, that $\lim_{x\to\infty}\big(f(x+1)-f(x)\big)=b$.

(b) Let $f:\mathbf{R}\to\mathbf{R}$ be differentiable in $\mathbf{R}$ with $f'(0)>0$ and f' continuous at 0. Show that there exists $\delta>0$ such that f is increasing in the interval $[-\delta,\delta]$.

Solution: Since f' is continuous at 0 and $f'(0)>0$, we can choose $\delta>0$ so that for all $x\in\mathbf{R}$

$$|x-0|\leq\delta\implies\big|f'(x)-f'(0)\big|\leq\frac{f'(0)}{2}\implies f'(x)\geq f'(0)-\frac{f'(0)}{2}=\frac{f'(0)}{2}>0.$$

Thus for all $x\in[-\delta,\delta]$ we have $f'(x)>0$, and so f is strictly increasing in the interval $[-\delta,\delta]$.

(c) Let $f:\mathbf{R}\to\mathbf{R}$ be continuous at 0 and differentiable in $\mathbf{R}\setminus\{0\}$ with $\lim_{x\to 0} f'(x)=b$. Show that f is differentiable at 0 with $f'(0)=b$.

Solution: Let $\epsilon>0$. Since $\lim_{x\to 0} f'(x)=b$ we can choose $\delta>0$ so that

$$0<|x-0|\leq\delta\implies\big|f'(x)-b\big|\leq\epsilon.$$

Let $x\in\mathbf{R}$ with $0<|x-a|\leq\delta$. Since f is differentiable in $(0,x]$ (or in $[x,0)$ in the case that $x<0$) and f is continuous at 0, we can invoke the Mean Value Theorem to choose a point c strictly between 0 and x so that $f'(c)=\frac{f(x)-f(0)}{x-0}$. Since $0<|x-a|\leq\delta$ and the point c lies between 0 and x , we also have $0<|c-0|\leq\delta$. Thus we have

$$\left|\frac{f(x)-f(0)}{x-0}-b\right|=\big|f'(c)-b\big|\leq\epsilon.$$

Thus we have $\lim_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=b$, that is $f'(0)=b$.