AMATH/PMATH 331 Real Analysis, Problems for Chapter 1

1: Let $\mathbf{F}=\mathbf{Q}$ or $\mathbf{R}$. For $a,b\in\mathbf{F}$ with $a\le b$ we write

$$(a,b)=\{x\in\mathbf{F}\,|\,a<x<b\}\ ,\ [a,b]=\{x\in\mathbf{F}\,|\,a\le x\le b\}\,,$$
$$(a,b]=\{x\in\mathbf{F}\,|\,a<x\le b\}\ ,\ [a,b)=\{x\in\mathbf{F}\,|\,a\le x<b\}\,.$$

A **bounded interval** in $\mathbf{F}$ is any set of one of the above forms. For a subset $A\subseteq\mathbf{F}$, we say that A has the **intermediate value property** when for every $a,b,x\in\mathbf{F}$ with $a<x<b$, if $a\in A$ and $b\in A$ then $x\in A$.

(a) Find a bounded set $A\subseteq\mathbf{Q}$ which has the intermediate value property but which is not a bounded interval in $\mathbf{Q}$.

(b) Show that for every bounded set $A\subseteq\mathbf{R}$, if A has the intermediate value property then A is a bounded interval in $\mathbf{R}$.

2: (a) Let $x_k=\frac{2k+1}{k-1}$ for $k\ge 2$. Use the definition of the limit to show that $\lim\limits_{k\to\infty}x_k=2$.

(b) Let $x_k=\frac{k}{\sqrt{k+3}}$ for $k\ge 0$. Use the definition of the limit to show that $\lim\limits_{n\to\infty}x_k=\infty$.

(c) Let $x_k=\sin(k)$ for $k\ge 0$. Use the definition of the limit to show that $(x_k)_{k\ge 0}$ diverges.

(d) Let $x_1=\frac{7}{2}$ and for $k\ge 1$ let $x_{k+1}=\dfrac{6}{5-x_k}$. Find $\lim\limits_{k\to\infty}x_k$ if it exists.

3: (a) Find a divergent sequence $(x_k)_{k\ge 0}$ in $\mathbf{R}$ with $\left|x_k-x_{k-1}\right|\le\frac{1}{k}$ for all $k\ge 1$.

(b) Let $(x_k)_{k\ge 0}$ be a sequence in $\mathbf{R}$ with $\left|x_k-x_{k-1}\right|\le\frac{1}{k^2}$ for all $k\ge 1$. Show that (x_k) converges in $\mathbf{R}$.

4: (a) Show that every sequence (x_k) in $\mathbf{R}$ has a monotonic subsequence. Hint: consider indices k with the property that $x_k>x_j$ for all $j>k$.

(b) Let $x_k=\frac{k}{\sqrt{2}}-\left\lfloor\frac{k}{\sqrt{2}}\right\rfloor$ for $k\ge 0$. Show that (x_k) has a monotonic subsequence (x_{k_j}) with $x_{k_j}\to 0$ as $j\to\infty$.

5: (a) Show that there exist (at least) 3 distinct values of x such that $8x^3=6x+1$.

(b) Let $f:[0,2]\to\mathbf{R}$ be continuous with $f(0)=f(2)$. Show that $f(x)=f(x+1)$ for some $x\in[0,1]$.

(c) Let $f:\mathbf{R}\to\mathbf{R}$ be continuous. Suppose that $|f(x)-f(y)|\ge|x-y|$ for all $x,y\in\mathbf{R}$. Show that f is surjective.

6: (a) Define $f,g:\mathbf{R}\to\mathbf{R}$ by $f(x)=x^3$ and $g(x)=\sqrt[3]{x}$. Show that g is uniformly continuous but that f is not.

(b) Find an example of a function $f:\mathbf{R}\to\mathbf{R}$ which is continuous and bounded but not uniformly continuous.

(c) Let $a,b\in\mathbf{R}$ with $a<b$, and let $f,g:[a,b]\to\mathbf{R}$. Suppose that f and g are both uniformly continuous and bounded. Show that fg is uniformly continuous.

7: (a) Let $f:\mathbf{R}\to\mathbf{R}$ be differentiable with $f(0)=3$. Suppose $f'(x)\le 1$ for all $x>0$. Prove that there is a number $a>0$ such that $f(a)=2a$.

(b) Let $f:\mathbf{R}\to\mathbf{R}$ be twice differentiable with $f(0)=0$ and $f(1)=1$ and $f'(0)=f'(1)=0$. Show that $|f''(x)|\ge 4$ for some $x\in[0,1]$.

(c) Prove that ${\sqrt{x}}^{\sqrt{x+1}}>{\sqrt{x+1}}^{\sqrt{x}}$ for all $x>e^2$.

8: (a) Let $f:\mathbf{R}\to\mathbf{R}$ be differentiable with $\lim\limits_{x\to\infty}f'(x)=b$. Show that $\lim\limits_{x\to\infty}\big(f(x+1)-f(x)\big)=b$.

(b) Let $f:\mathbf{R}\to\mathbf{R}$ be differentiable in $\mathbf{R}$ with $f'(0)>0$ and f' continuous at 0. Show that there exists $\delta>0$ such that f is increasing in the interval $[-\delta,\delta]$.

(c) Let $f:\mathbf{R}\to\mathbf{R}$ be continuous at 0 and differentiable in $\mathbf{R}\setminus\{0\}$ with $\lim\limits_{x\to 0}f'(x)=b$. Show that f is differentiable at 0 with $f'(0)=b$.