Chapter 8. Fourier Series

Fourier Series

8.1 Definition: A (real-valued) **trigonometric polynomial** is a function $p:\mathbb{R}\to\mathbb{R}$ of the form

$$p(x)=a_0+\sum_{n=1}^{m}a_n\cos nx+\sum_{n=1}^{m}b_n\sin nx$$

for some $a_n,b_n\in\mathbb{R}$, and we say that $p(x)$ is of degree m when either $a_m\neq 0$ or $b_m\neq 0$. A (real-valued) **trigonometric series** is a series of the form

$$s(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

which denotes the sequence of partial sums $s_m(x)=a_0+\sum_{n=1}^{m}a_n\cos nx+\sum_{n=1}^{m}b_n\sin nx$.

8.2 Note: Every trigonometric polynomial is a smooth 2π-periodic function $p:\mathbb{R}\to\mathbb{R}$. If a trigonometric series converges pointwise to a function $f:\mathbb{R}\to\mathbb{R}$, then f must be 2π-periodic (but not necessarily continuous). Every 2π-periodic function $f:\mathbb{R}\to\mathbb{R}$ determines, and is determined by, its restriction $f:[-\pi,\pi]\to\mathbb{R}$ with $f(-\pi)=f(\pi)$. When S is the unit circle $S=\big\{z\in\mathbb{C}\big||z|=1\big\}$, every 2π-periodic function $f:\mathbb{R}\to\mathbb{R}$ determines, and is determined by, the function $g:S\to\mathbb{R}$ given by $g(e^{i\,x})=f(x)$.

8.3 Note: Recall the following trigonometric identities: for $n,m\in\mathbb{Z}$ we have

$$\begin{aligned}
\sin^2 nx &= \tfrac{1}{2}\big(1-\cos 2nx\big)\\
\cos^2 nx &= \tfrac{1}{2}\big(1+\cos 2nx\big)\\
\sin nx\,\cos nx &= \tfrac{1}{2}\,\sin 2nx\\
\sin nx\sin mx &= \tfrac{1}{2}\big(\cos(n-m)x-\cos(n+m)x\big)\\
\cos nx\cos mx &= \tfrac{1}{2}\big(\cos(n-m)x+\cos(n+m)x\big)\\
\sin nx\,\cos mx &= \tfrac{1}{2}\big(\sin(n+m)x+\sin(n-m)x\big)
\end{aligned}$$

8.4 Theorem: *(Orthogonality Relations) For $n,m\in\mathbb{Z}^+$ we have*

$$\int_{-\pi}^{\pi}1=2\pi\ ,\ \int_{-\pi}^{\pi}\cos^2 nx\,dx=\pi\ ,\ \int_{-\pi}^{\pi}\sin^2 nx\,dx=\pi,$$

$$\int_{-\pi}^{\pi}\cos nx\,dx=\int_{-\pi}^{\pi}\sin nx\,dx=\int_{-\pi}^{\pi}\cos nx\,\sin mx\,dx=0\ ,\ \textit{and}$$

$$\textit{if } n\neq m \textit{ then }\int_{-\pi}^{\pi}\cos nx\,\cos mx\,dx=\int_{-\pi}^{\pi}\sin nx\,\sin mx\,dx=0.$$

Proof: These follow from the trigonometric identities in Note 8.3.

8.5 Corollary: *The set $\left\{\frac{1}{\sqrt{2\pi}}\,,\,\frac{1}{\sqrt{\pi}}\cos nx\,,\,\frac{1}{\sqrt{\pi}}\sin nx\,\middle|\,n\in\mathbb{Z}^+\right\}$ is an orthonormal set in the inner product space $\mathcal{C}[-\pi,\pi]$ using its standard inner product given by*

$$\langle f,g\rangle=\int_{-\pi}^{\pi}f(x)g(x)\,dx.$$

8.6 Corollary: *When* $p(x)=a_0+\sum_{n=1}^{m}a_n\cos nx+\sum_{n=1}^{m}b_n\sin nx$*, the coefficients are given by*

$$\begin{aligned} a_0 &= \tfrac{1}{2\pi}\langle p,1\rangle = \tfrac{1}{2\pi}\int_{-\pi}^{\pi}p(x)\,dx\,, \\ a_n &= \tfrac{1}{\pi}\langle p,\cos nx\rangle = \tfrac{1}{\pi}\int_{-\pi}^{\pi}p(x)\cos nx\;dx \\ b_n &= \tfrac{1}{\pi}\langle p,\sin nx\rangle = \tfrac{1}{\pi}\int_{-\pi}^{\pi}p(x)\sin nx\;dx\,. \end{aligned}$$

8.7 Definition: When $f:[-\pi,\pi]\to\mathbb{R}$ is Riemann integrable (or when $f:\mathbb{R}\to\mathbb{R}$ is a 2π-periodic function whose restriction to $[-\pi,\pi]$ is Riemann integrable), we define the (real) **Fourier coefficients** of f to be the real numbers

$$\begin{aligned} a_0(f) &= \tfrac{1}{2\pi}\langle f,1\rangle = \tfrac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx \\ a_n(f) &= \tfrac{1}{\pi}\langle f,\cos nx\rangle = \tfrac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\;dx \\ b_n(f) &= \tfrac{1}{\pi}\langle f,\sin nx\rangle = \tfrac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\;dx \end{aligned}$$

we define the m^{th} **Fourier polynomial** of f to be the trigonometric polynomial

$$s_m(f)(x)=a_0(f)+\sum_{n=1}^{m}a_n(f)\cos nx+\sum_{n=1}^{m}b_n(f)\sin nx$$

and we define the (real) **Fourier series** of f to be the trigonometric series

$$s(f)(x)=a_0(f)+\sum_{n=1}^{\infty}a_n(f)\cos nx+\sum_{n=1}^{\infty}b_n(f)\sin nx.$$

Note that $s(f)(x)$ can denote either the sequence of partial sums $\big(s_m(f)(x)\big)_{m\geq 0}$ (which may or may not converge), or its sum (when it does converges).

8.8 Example: Find the Fourier series of the 2π-periodic function $f:\mathbb{R}\to\mathbb{R}$ with

$$f(x)=\begin{cases}\frac{\pi}{2}+x \text{ for } -\pi\leq x\leq 0,\\ \frac{\pi}{2}-x \text{ for } \;0\leq x\leq \pi.\end{cases}$$

Solution: Since $f(x)$ is even, we have $b_n=0$ for all $n\in\mathbb{Z}^+$, and we have

$$\begin{aligned} a_0 &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx = \tfrac{1}{\pi}\int_0^{\pi}f(x)\,dx = \tfrac{1}{\pi}\int_0^{\pi}\tfrac{\pi}{2}-x\;dx = \tfrac{1}{\pi}\Big[\tfrac{\pi}{2}x-\tfrac{1}{2}x^2\Big]_0^{\pi}=0 \\ a_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\;dx = \tfrac{2}{\pi}\int_0^{\pi}f(x)\cos nx\;dx = \tfrac{2}{\pi}\int_0^{\pi}\big(\tfrac{\pi}{2}-x\big)\cos nx\;dx \\ &= \int_0^{\pi}\cos nx\;dx-\tfrac{2}{\pi}\int_0^{\pi}x\,\cos nx\;dx = \Big[\tfrac{1}{n}\sin nx\Big]_0^{\pi}-\tfrac{2}{\pi}\Big[\tfrac{1}{n}x\,\sin nx+\tfrac{1}{n^2}\cos nx\Big]_0^{\pi} \\ &= 0-\tfrac{2}{\pi}\tfrac{1}{n^2}\big((-1)^n-1\big)=\begin{cases}0 & \text{if } n \text{ is even,}\\ \frac{4}{\pi\,n^2} & \text{if } n \text{ is odd.}\end{cases} \end{aligned}$$

Thus we have

$$s(f)(x)=\tfrac{4}{\pi}\big(\tfrac{1}{1^2}\cos x+\tfrac{1}{3^2}\cos 3x+\tfrac{1}{5^2}\cos 5x+\cdots\big)$$

8.9 Remark: Fourier series can be used as a useful tool to solve many differential equations. Some examples of such applications are presented in Appendix 1.

8.10 Remark: The difference between the terms "Fourier series" and "trigonometric series" is the same as the difference between the terms "Taylor series" and "powers series": not every trigonometric series is a Fourier series; a Fourier series is a trigonometric series which is obtained from some function $f : [-\pi, \pi] \to \mathbb{R}$ as in Definition 8.7.

8.11 Remark: There are many questions we can ask about convergence of Fourier series, and some of these questions are surprisingly difficult to answer. For example, we can ask the following questions: Under what conditions on the coefficients a_n and b_n does the corresponding trigonometric series converge? Can we have two different trigonometric series which converge to the same function? Given a 2π-periodic Riemann integrable (or continuous) function f, does the Fourier series of f always converge? If the Fourier series of f does converge, then does it necessarily converge to f? And each of these convergence questions splits to several different questions depending on which kind of convergence we are considering (pointwise convergence, uniform convergence, or convergence using one of the p-norms).

8.12 Example: Consider the function $f(x)$ from Example 8.8. The Fourier series $s(f)$ given by

$$s(f)(x) = \tfrac{4}{\pi}\big(\tfrac{1}{1^2}\cos x + \tfrac{1}{3^2}\cos 3x + \tfrac{1}{5^2}\cos 5x + \cdots\big)$$

converges uniformly (on $\mathbb{R}$) by the Weierstrass M-Test, so the sum of the Fourier series, also denoted by $s(f)$, is a well-defined continuous function $s(f) : \mathbb{R} \to \mathbb{R}$. If we knew that the series $s(f)$ converges pointwise to f (so that $s(f)(x) = f(x)$ for all x) then, in particular, we would have $s(f)(0) = f(0)$, that is $\frac{4}{\pi}\big(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\big) = \frac{\pi}{2}$, so we would obtain the formula

$$\sum_{k=1}^{\infty} \tfrac{1}{(2k+1)^2} = \tfrac{\pi^2}{8}.$$

We shall do this in Example 8.40, at the end of the Chapter.

Semi Inner Product Spaces and Semi Normed Linear Spaces

8.13 Definition: A **semi inner product** on a vector space U is a function $\langle\ ,\ \rangle : U\times U \to \mathbb{R}$ such that for all $u, v, w \in U$ and all $t \in \mathbb{R}$ we have

(1) (Bilinearity) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle\ ,\ \langle tu, v\rangle = t\,\langle u, v\rangle$,
$\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle\ ,\ \langle u, tv\rangle = t\,\langle u, v\rangle$,
(2) (Symmetry) $\langle u, v\rangle = \langle v, u\rangle$, and
(3) (Positive Semi-Definiteness) $\langle u, u\rangle \geq 0$.

A **semi inner product space** is a vector space with a semi inner product.

8.14 Definition: A **semi norm** on a vector space U is a function $\|\ \| : U \to \mathbb{R}$ such that for all $u, v \in U$ and all $t \in \mathbb{R}$ we have

(1) (Scaling) $\|tu\| = |t|\,\|u\|$,
(2) (Positive Semi-Definiteness) $\|u\| \geq 0$, and
(3) (Triangle Inequality) $\|u+v\| \leq \|u\| + \|v\|$.

A **semi normed linear space** is a vector space equipped with a semi norm.

8.15 Definition: Let U be a semi normed linear space. When $a \in U$ and $(x_n)_{n\geq 1}$ is a sequence in U we say that (x_n) **converges** to a in U and write $x_n \to a$ in U when for all $\epsilon > 0$ there exists $m \in \mathbb{Z}^+$ such that $n \geq m \Longrightarrow \|x_n - a\| < \epsilon$. When $A \subseteq U$, we say that A is **dense** in U when for every $u \in U$ and every $\epsilon > 0$, there exists $a \in A$ with $\|u - a\| < \epsilon$.

8.16 Example: Recall that $\mathcal{B}[-\pi,\pi] = \big\{f : [-\pi,\pi] \to \mathbb{R}\,\big|\, f \text{ is bounded}\big\}$. Let

$$\begin{aligned}
\mathcal{B}(T) &= \Big\{f \in \mathcal{B}[-\pi,\pi]\,\Big|\, f(-\pi) = f(\pi)\Big\}\\
\mathcal{R}(T) &= \Big\{f \in \mathcal{B}(T)\,\Big|\, f \text{ is Riemann integrable}\Big\}\\
\mathcal{C}(T) &= \Big\{f \in \mathcal{B}(T)\,\Big|\, f \text{ is continuous}\Big\}\\
\mathcal{P}(T) &= \Big\{f \in \mathcal{C}(T)\,\Big|\, f \text{ is (the restriction of) a trigonometric polynomial}\Big\}
\end{aligned}$$

These spaces can all be identified with corresponding spaces of 2π-periodic functions.

The space $\mathcal{B}(T)$ $\big($hence also each of the spaces $\mathcal{R}(T)$, $\mathcal{C}(T)$ and $\mathcal{P}(T)\big)$ is a normed linear space under the supremum norm given by

$$\|f\|_\infty = \max\Big\{|f(x)|\,\Big|\, x \in [-\pi,\pi]\Big\}.$$

For $f, g \in \mathcal{R}(T)$, we define

$$\begin{aligned}
\langle f, g\rangle &= \int_{-\pi}^{\pi} f(x)g(x)\,dx\,,\\
\|f\|_1 &= \int_{-\pi}^{\pi} |f(x)|\,dx\,,\\
\|f\|_2 &= \left(\int_{-\pi}^{\pi} |f(x)|^2\,dx\right)^{1/2}.
\end{aligned}$$

Note that $\langle f, g\rangle$ does not quite define an inner product on $\mathcal{R}(T)$ and $\|f\|_1$ and $\|f\|_2$ do not quite define norms on $\mathcal{R}(T)$ because they do not satisfy the positive definite property. But $\langle f, g\rangle$ is a semi inner product on $\mathcal{R}(T)$ and $\|f\|_1$ and $\|f\|_2$ are semi norms on $\mathcal{R}(T)$. On the other hand, when we restrict these to $\mathcal{C}(T)$, $\langle f, g\rangle$ does give an inner product and $\|f\|_1$ and $\|f\|_2$ do give norms on $\mathcal{C}(T)$ $\big($hence also on $\mathcal{P}(T)\big)$.

8.17 Theorem: *(Properties of Semi Inner Products and Semi Norms) Let U be a semi inner product space, and define $\|\ \ \| : U \to \mathbb{R}$ by $\|u\| = \sqrt{\langle u, u\rangle}$. Then $\|\ \ \|$ is a semi norm on U. Indeed for $u, v \in U$ and $t \in \mathbb{R}$ we have*

(1) (Scaling) $\|tu\| = |t|\,\|u\|$,
(2) (Positive Semi-Definiteness) $\|u\| \geq 0$,
(3) $\|u \pm v\|^2 = \|u\|^2 \pm 2\,\langle u, v\rangle + \|v\|^2$,
(4) (Pythagoras' Theorem) $\langle u, v\rangle = 0 \iff \|u+v\|^2 = \|u\|^2 + \|v\|^2$,
(5) (Parallelogram Law) $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$,
(6) (Polarization Identity) $\langle u, v\rangle = \frac{1}{4}\big(\|u+v\|^2 - \|u-v\|^2\big)$,
(7) (The Cauchy-Schwarz Inequality) $\big|\langle u, v\rangle\big| \leq \|u\|\,\|v\|$, and
(8) (The Triangle Inequality) $\big|\|u\| - \|v\|\big| \leq \|u+v\| \leq \|u\| + \|v\|$.

Proof: The proof is left as an exercise (it is almost identical to the proof of Theorem 4.7).

8.18 Theorem: *(Orthogonal Projection) Let W be a semi inner product space, let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal set in W, $\big($meaning that $\langle e_k, e_k\rangle = 1$ for all k and $\langle e_k, e_\ell\rangle = 0$ for all $k \neq \ell\big)$ and let $U = \mathrm{Span}\{e_1, e_2, \cdots, e_n\}$. Then*

(1) If $u = \sum\limits_{k=1}^{n} a_k e_k \in U$ then $\|u\|^2 = \sum\limits_{k=1}^{n} {a_k}^2$. In particular, U is an inner product space.

(2) If $u = \sum\limits_{k=1}^{n} a_k e_k \in U$ then $a_k = \langle u, e_k\rangle$ for all indices k.

(3) If $x \in W$ and $u = \mathrm{Proj}_U(x) = \sum\limits_{k=1}^{n} \langle x, e_k\rangle e_k \in U$, then $\langle x-u, v\rangle = 0$ for all $v \in U$.

(4) If $x \in W$ then $u = \mathrm{Proj}_U(x) = \sum\limits_{k=1}^{n} \langle x, e_k\rangle e_k \in U$ is the unique point in U with $\|x-u\| \leq \|x-w\|$ for every $w \in U$. We say u is the **orthogonal projection** *of x onto U.*

Proof: To prove Parts 1 and 2, let $u = \sum\limits_{k=1}^{n} a_k e_k \in U$. Then

$$\|u\|^2 = \langle u, u\rangle = \Big\langle \sum_{k=1}^{n} a_k e_k, \sum_{\ell=1}^{n} a_\ell e_\ell \Big\rangle = \sum_{k=1}^{n}\sum_{\ell=1}^{n} a_k a_\ell \langle e_k, e_\ell\rangle = \sum_{k=1}^{n} {a_k}^2.$$

It follows that when the semi inner product is restricted to U it becomes positive definite because $\langle u, u\rangle = 0 \Longrightarrow \sum\limits_{k=1}^{n} {a_k}^2 = 0 \Longrightarrow$ each $a_k = 0 \Longrightarrow u = 0$. Also, for each index k

$$\langle u, e_k\rangle = \Big\langle \sum_{j=1}^{n} a_j e_j, e_k \Big\rangle = \sum_{j=1}^{n} a_j \langle e_j, e_k\rangle = a_k.$$

To prove Parts 3 and 4, let $x \in W$, let $u = \sum\limits_{k=1}^{n} \langle x, e_k\rangle e_k \in U$, let $v = \sum\limits_{k=1}^{n} a_k e_k \in U$ and let $w \in U$. By Part 1, we have $\langle x, e_k\rangle = \langle u, e_k\rangle$ for all k, and so

$$\langle x, v\rangle = \Big\langle x, \sum_{k=1}^{n} a_k e_k \Big\rangle = \sum_{k=1}^{n} a_k \langle x, e_k\rangle = \sum_{k=1}^{n} a_k \langle u, e_k\rangle = \Big\langle u, \sum_{k=1}^{n} a_k e_k \Big\rangle = \langle u, v\rangle$$

hence $\langle x-u, v\rangle = \langle x, v\rangle - \langle u, v\rangle = 0$, proving Part 3. Also, since $u \in U$ and $w \in U$ we have $w-u \in U$, and so by Part 3 (with $v = w-u$) we have $\langle x-u, w-u\rangle = 0$. By Pythagoras' Theorem,

$$\|x-w\|^2 = \big\|(x-u) + (u-w)\big\|^2 = \|x-u\|^2 + \|u-w\|^2 \geq \|x-u\|^2.$$

To see that u is unique, note that if we had $w \in U$ with $\|x-w\| = \|x-u\|$ then $\big($since $\|x-w\|^2 = \|x-u\|^2 + \|u-w\|^2\big)$ we would have $\|u-w\| = 0$ and hence $u = w$ by Part 1.

Density of Trigonometric Polynomials

8.19 Theorem: *$\mathcal{P}(T)$ is dense in $\big(\mathcal{C}(T), \|\ \ \|_\infty\big)$.*

Proof: First we note that the space $\mathcal{P}(T)$ is a subalgebra of $\mathcal{C}(T)$. Indeed, it is easy to see that when $p, q \in \mathcal{P}(T)$ and $c \in \mathbb{R}$ we have $cp \in \mathcal{P}(T)$ and $p+q \in \mathcal{P}(T)$, and we also have $pq \in \mathcal{P}(T)$ by the trigonometric identities listed in Note 8.4.

Let S be the unit circle $S = \big\{e^{ix}\,\big|\, x \in [-\pi, \pi)\big\}$. Let $\phi : \mathcal{C}(T) \to \mathcal{C}(S)$ be the natural correspondence given by $\phi(f) = u$ where $u(e^{ix}) = f(x)$. Note that ϕ is linear and ϕ preserves product, that is $\phi(f+g) = \phi(f)+\phi(g)$ and $\phi(cf) = c\phi(f)$ and $\phi(fg) = \phi(f)\phi(g)$ when $c \in \mathbb{R}$ and $f, g \in \mathcal{C}(T)$. It follows that ϕ sends subalgebras of $\mathcal{C}(T)$ to subalgebras of $\mathcal{C}(S)$. Let $\mathcal{P}(S) = \phi\big(\mathcal{P}(T)\big) = \big\{\phi(p)\,\big|\, p \in \mathcal{P}(T)\big\}$, and note that $\mathcal{P}(T)$ is a subalgebra of $\mathcal{C}(T)$ and $\mathcal{P}(S)$ is the corresponding subalgebra of $\mathcal{C}(S)$. Also note that ϕ (hence also ϕ^{-1} preserves norm, that is $\|\phi(f)\|_\infty = \|f\|_\infty$, so ϕ is a homeomorphism. It follows that ϕ gives a bijective correspondence between the open (or closed) subsets of $\mathcal{C}(T)$ and the open (or closed) subsets of $\mathcal{C}(S)$ and hence when $A \subseteq (T)$, we have $\overline{\phi(A)} = \phi\big(\overline{A}\big)$.

Note that ϕ sends the constant function 1, which lies in $\mathcal{P}(T)$, to the constant function 1 in $\mathcal{P}(S)$, so the subalgebra $\mathcal{P}(S)$ vanishes nowhere. Also note that ϕ sends the functions $p(x) = \cos x$ and $q(x) = \sin x$, which both lie in $\mathcal{P}(T)$, to the functions $\phi(p)(e^{ix}) = \cos x$ and $\phi(q)(e^{ix}) = \sin x$ in $\mathcal{P}(S)$. It follows that the subalgebra $\mathcal{P}(S)$ separates points (given $z, w \in S$ with $z \neq w$, say $z = e^{ix}$ and $w = e^{iy}$ where $x, y \in [-\pi, \pi)$ with $x \neq y$, since $(\cos x, \sin x) = z \neq w = (\cos y, \sin y)$ it follows that either $\cos x \neq \cos y$ or $\sin x \neq \sin y$).

Since S is compact (it is a closed, bounded subset of $\mathbb{R}^2$) and $\mathcal{P}(S)$ separates points and vanishes nowhere, it follows from the Stone-Weierstrass Theorem that $\mathcal{P}(S)$ is dense in $\mathcal{C}(S)$, that is $\overline{\mathcal{P}(S)} = \mathcal{C}(S)$, using the supremum metric. Since ϕ is a homeomorphism, it follows that $\mathcal{P}(T)$ is dense in $\mathcal{C}(T)$, that is $\overline{\mathcal{P}(T)} = \mathcal{C}(T)$, using the supremum metric.

8.20 Theorem: *$\mathcal{P}(T)$ is dense in $\big(\mathcal{R}(T), \|\ \ \|_1\big)$.*

Proof: Choose a partition $X = (x_0, x_1, \cdots, x_n)$ of $[-\pi, \pi]$ so that $\displaystyle\int_{-\pi}^{\pi} f(x)\,dx - L(f, X) < \frac{\epsilon}{3}$ where $L(f, X)$ is the lower Riemann sum given by $L(f, X) = \sum\limits_{k=1}^{n} m_k(x_k - x_{k-1})$ with $m_k = \inf\big\{|f(x)|\,\big|\, x \in [x_{k-1}, x_k]\big\}$. Let $s : [-\pi, \pi] \to \mathbb{R}$ be the step function given by $s(x) = m_k$ when $x \in [x_{k-1}, x_k)$ and $s(\pi) = m_n$. Then $s(x) \leq f(x)$ for all $x \in [-\pi, \pi]$ and we have $\displaystyle\int_{-\pi}^{\pi} s(x)\,dx = L(f, X)$. Choose a piecewise linear continuous function $\ell : [-\pi, \pi] \to \mathbb{R}$ with $\ell(-\pi) = \ell(\pi)$ so that $\displaystyle\int_{-\pi}^{\pi} \big|s(x) - \ell(x)\big|\,dx < \frac{\epsilon}{3}$, for example define $\ell(x)$ on the interval $[x_{k-1}, x_k]$ by $\ell(x) = \frac{m_k}{r}(x - x_{k-1})$ when $x_{k-1} \leq x \leq x_{k-1}+r$, and $\ell(x) = m_k = s(x)$ when $x_{k-1}+r \leq x \leq x_k-r$, and $\ell(x) = -\frac{m_k}{r}(x - x_k)$ when $x_k - r \leq x \leq x_k$, with $r < \frac{x_k - x_{k-1}}{2}$ and $r < \frac{\epsilon}{3m}$ where $m = \sum\limits_{k=1}^{n} m_k$ so that $\displaystyle\int_{-\pi}^{\pi} |s(x) - \ell(x)|\,dx = \sum_{k=1}^{n} r m_k < \frac{\epsilon}{3}$. Finally, by the above theorem, we can choose a trigonometric polynomial $p \in \mathcal{P}(T)$ such that $\|p - \ell\|_\infty < \frac{\epsilon}{6\pi}$. Then we have

$$\int_{-\pi}^{\pi} \big|f(x) - p(x)\big|\,dx \leq \int_{-\pi}^{\pi} |f(x) - s(x)| + |s(x) - \ell(x)| + |\ell(x) - p(x)|\ dx < \epsilon$$

because $\int_{-\pi}^{\pi} |f(x) - s(x)|\,dx = \int_{-\pi}^{\pi} f(x) - s(x)\ dx = \int_{-\pi}^{\pi} f(x)\,dx - L(f, X) < \frac{\epsilon}{3}$, and $\int_{-\pi}^{\pi} |s(x) - \ell(x)|\,dx < \frac{\epsilon}{3}$, and $\int_{-\pi}^{\pi} \big|\ell(x) - p(x)\big|\,dx \leq \int_{-\pi}^{\pi} \|\ell - p\|_\infty\,dx = 2\pi\|\ell - p\|_\infty < \frac{\epsilon}{3}$.

8.21 Theorem: *$\mathcal{P}(T)$ is dense in $\big(\mathcal{R}(T), \|\ \ \|_2\big)$.*

Proof: Let $\epsilon > 0$. We begin by constructing a step function $s : [-\pi, \pi] \to \mathbb{R}$ such that $\|f - s\|_2 < \frac{\epsilon}{3}$. Let $M = \|f\|_\infty = \sup\big\{|f(x)|\,\big|\, x \in [-\pi, \pi]\big\}$. Choose a partition $X = (x_0, x_1, \cdots, x_n)$ such that

$$U(f,X) - L(f,X) = \sum_{k=1}^n (M_k - m_k)\Delta_k x < \min\left\{\tfrac{\epsilon^2}{18}, \tfrac{\epsilon^2}{18(2M)^2}\right\}$$

where $M_k = \sup\big\{|f(x)|\,\big|\, x \in [x_{k-1}, x_k]\big\}$ and $m_k = \inf\big\{|f(x)|\,\big|\, x \in [x_{k-1}, x_k]\big\}$. Let $s : [\pi, \pi] \to \mathbb{R}$ be the step function given by $s(x) = m_k$ when $x \in [x_{k-1}, x_k)$ and $s(\pi) = m_n$ so that $s(x) \leq f(x)$ for all $x \in [-\pi, \pi]$ and $\displaystyle\int_{-\pi}^{\pi} s(x)\,dx = \sum_{k=1}^n m_k\,\Delta_k x = L(f,X)$. Let $K = \big\{k \in \{1, 2, \cdots, n\}\,\big|\, M_k - m_k > 1\big\}$ and $L = \big\{k \in \{1, 2, \cdots, n\}\,\big|\, M_k - m_k \leq 1\big\}$. Note that

$$\sum_{k\in K} \Delta_k x \leq \sum_{k\in K} (M_k - m_k)\Delta_k x \leq \sum_{k=1}^n (M_k - m_k)\Delta_k x \tfrac{\epsilon^2}{18M}.$$

Also note that since $M = \|f\|_\infty$, we have $M_k \leq M$ and $m_k \geq -M$ so that $(M_k - m_k) \leq 2M$ for all indices k. Thus

$$\begin{aligned}\int_{-\pi}^{\pi} \big|f(x) - s(x)\big|^2 dx &= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \big|f(x) - m_k\big|^2\,dx \leq \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \big|M_k - m_k\big|^2\,dx \\ &= \sum_{k=1}^n (M_k - m_k)^2 \Delta_k x = \sum_{k\in K} (M_k - m_k)^2 \Delta_k x + \sum_{k \in L} (M_k - m_k)^2 \Delta_k x \\ &\leq \sum_{k\in K} (2M)^2 \Delta_k x + \sum_{k\in L} (M_k - m_k)\,\Delta_k x \leq (2M)^2 \sum_{k\in K} \Delta_k x + \sum_{k=1}^n (M_k - m_k)\Delta_k \\ &< \tfrac{\epsilon^2}{18} + \tfrac{\epsilon^2}{18} = \tfrac{\epsilon^2}{9}\end{aligned}$$

so that $\|f - s\|_2 < \frac{\epsilon}{3}$, as required.

Next, we construct a continuous piecewise-linear function $\ell : [-\pi, \pi] \to \mathbb{R}$ with $\ell(-\pi) = \ell(\pi)$ such that $\|s - \ell\|_2 < \frac{\epsilon}{3}$. We can take ℓ to be given on the interval $[x_{k-1}, x_k]$ by $\ell(x) = \frac{m_k}{r}(x - x_{k-1})$ when $x_{k-1} \leq x \leq x_{k-1} + r$, and $\ell(x) = m_k$ when $x_{k-1} + r \leq x \leq x_k - r$, and $\ell(x) = -\frac{m_k}{r}(x - x_k)$ when $x_k - r \leq x \leq x_k$ where $r > 0$ is chosen with $r < \frac{x_k - x_{k-1}}{2}$ for all k and with $r < \frac{\epsilon^2}{6m}$ where $m = \sum {m_k}^2$. Then we have

$$\begin{aligned}\int_{-\pi}^{\pi} \big|s(x) - \ell(x)\big|^2\,dx &= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \big|m_k - \ell(x)\big|^2\,dx = \sum_{k=1}^n 2\int_0^r \left(\tfrac{m_k}{r}\,x\right)^2 dx \\ &= \sum_{k=1}^n 2\tfrac{{m_k}^2 r}{3} = \tfrac{2m}{3} r < \tfrac{\epsilon^2}{9}\end{aligned}$$

so that $\|s - \ell\|_2 < \frac{\epsilon}{3}$, as required.

Finally, by Theorem 8.14 we can choose $p \in \mathcal{P}(T)$ such that $\|\ell - p\|_\infty < \frac{\epsilon^2}{18\pi}$ and then

$$\int_{-\pi}^{\pi} \big|\ell(x) - p(x)\big|^2\,dx \leq \int_{-\pi}^{\pi} {\|\ell - p\|_\infty}^2\,dx = 2\pi {\|\ell - p\|_\infty}^2 < \tfrac{\epsilon}{9}$$

so that $\|\ell - p\|_2 < \frac{\epsilon}{3}$. Thus

$$\|f - p\|_2 \leq \|f - s\|_2 + \|s - \ell\|_2 + \|\ell - p\|_2 < \epsilon.$$

8.22 Theorem: *(The Riemann-Lebesgue Lemma) Let $f \in \mathcal{R}(T)$. Then $a_n(f) \to 0$ and $b_n(f) \to 0$ in $\mathbb{R}$ as $n \to \infty$.*

Proof: Let $\epsilon > 0$. Since $\mathcal{P}(T)$ is dense in $\big(\mathcal{R}(T), \|\ \ \|_1\big)$, we can choose $p \in \mathcal{P}(T)$ such that $\displaystyle\int_{-\pi}^{\pi} \big|f(x) - p(x)\big|\,dx < \epsilon$. Let m be the degree of p and note that $a_n(p) = b_n(p) = 0$ for all $n > m$. So for $n > m$ we have

$$\big|a_n(f)\big| = \big|a_n(f) - a_n(p)\big| = \left|\tfrac{1}{\pi}\int_{-\pi}^{\pi} \big(f(x) - p(x)\big)\cos nx\,dx\right| \le \tfrac{1}{\pi}\int_{-\pi}^{\pi}\big|f(x) - p(x)\big|\,dx < \epsilon.$$

This shows that $a_n(f) \to 0$ and a similar calculation shows that $b_n(f) \to 0$.

8.23 Theorem: *Let $f, g \in \mathcal{R}(T)$. Then*

(1) (Polynomial Approximation) $s_m(f)$ is the unique trigonometric polynomial of degree at most m which is nearest to f in $\big(\mathcal{R}(T), \|\ \ \|_2\big)$.

(2) (Convergence) $s_m(f) \to f$ in $\big(\mathcal{R}(T), \|\ \ \|_2\big)$.

(3) (Parseval's Identity) $\|f\|_2{}^2 = 2\pi\, a_0(f)^2 + \pi \sum\limits_{n=1}^{\infty} a_n(f)^2 + \pi \sum\limits_{n=1}^{\infty} b_n(f)^2$.

(4) (Inner Product Formula) $\langle f, g\rangle = 2\pi\, a_0(f)a_0(g) + \pi \sum\limits_{n=1}^{\infty} a_n(f)a_n(g) + \pi \sum\limits_{n=1}^{\infty} b_n(f)b_n(g)$.

Proof: Part 1 of this Theorem follows from Part 4 of the Orthogonal Projection Theorem. Indeed the set

$$\left\{ \tfrac{1}{\sqrt{2\pi}}\,,\ \tfrac{1}{\sqrt{\pi}}\cos n\pi\,,\ \tfrac{1}{\sqrt{\pi}}\sin n\pi \,\middle|\, 1 \le n \le m \right\}$$

is an orthonormal set in $\mathcal{R}(T)$ which spans the subspace $\mathcal{P}_m(T)$ of trigonometric polynomials of degree at most m, and (as you can verify as an exercise using Definition 8.7) $s_m(f)$ is the orthogonal projection of f onto $\mathcal{P}_m(T)$.

To prove Part 2, let $\epsilon > 0$. Since $\mathcal{P}(T)$ is dense in $\big(\mathcal{R}(T)), \|\ \ \|_2\big)$, we can choose $p \in \mathcal{P}(T)$ such that $\|f - p\|_2 < \epsilon$. Let $d = \deg(p)$. For every $m > d$, since $s_m(f)$ is the point in $\mathcal{P}_m(T)$ nearest to f, and $p \in \mathcal{P}_d(T) \subseteq \mathcal{P}_m(T)$, we have $\|f - s_m(f)\|_2 \le \|f - p\|_2 < \epsilon$. Thus $s_m(f) \to f$ in $\big(\mathcal{R}(T), \|\ \ \|_2\big)$.

By Part 3 of the Orthogonal Projection Theorem, we have $\langle f - s_m(f), s_m(f)\rangle = 0$ so that $\langle f, s_m(f)\rangle = \langle s_m(f), s_m(f)\rangle = \|s_m(f)\|_2{}^2$, and so

$$\|f - s_m(f)\|_2{}^2 = \|f\|_2{}^2 - 2\langle f, s_m(f)\rangle + \|s_m(f)\|_2{}^2 = \|f\|_2{}^2 - \|s_m(f)\|_2{}^2.$$

Since $\|f\|_2{}^2 - \|s_m(f)\|_2{}^2 = \|f - s_m(f)\|_2{}^2 \to 0$, it follows that $\|s_m(f)\|_2{}^2 \to \|f\|_2{}^2$ in $\mathbb{R}$. Verify, as an exercise, that $\|s_m(f)\|_2{}^2 = 2\pi\, a_0(f)^2 + \pi \sum\limits_{n=1}^{m} a_n(f)^2 + \pi \sum\limits_{n=1}^{m} b_n(f)^2$, and so

$$\begin{aligned}
\|f\|_2{}^2 &= \lim_{m\to\infty} \|s_m(f)\|_2{}^2 \\
&= \lim_{m\to\infty} \Big(2\pi\, a_0(f)^2 + \pi \sum_{n=1}^{m} a_n(f)^2 + \pi \sum_{n=1}^{m} b_n(f)^2\Big) \\
&= 2\pi\, a_0(f)^2 + \pi \sum_{n=1}^{\infty} a_n(f)^2 + \pi \sum_{n=1}^{\infty} b_n(f)^2.
\end{aligned}$$

We leave it as an exercise to prove Part 4 using the Polarization Identity.

8.24 Remark: Note that for $f \in \mathcal{R}(T)$, Parseval's Identity is stronger than The Riemann-Lebesgue Lemma. The Riemann-Lebesgue Lemma states that $a_n(f) \to 0$ and $b_n(f) \to 0$, but Parseval's Identity implies that $\sum a_n(f)^2$ and $\sum b_n(f)^2$ both converge.

Complex-Valued Trigonometric Series

8.25 Definition: For a function $f:[a,b]\subseteq\mathbb{R}\to\mathbb{C}$ given by $f(x)=u(x)+i\,v(x)$ where $u,v:[a,b]\to\mathbb{R}$, we say that f is **Riemann integrable** on $[a,b]$ if and only if both u and v are Riemann integrable on $[a,b]$ and, in this case, we define

$$\int_a^b f(x)\,dx=\int_a^b u(x)\,dx+i\int_a^b v(x)\,dx.$$

8.26 Note: Given $a_n,b_n\in\mathbb{R}$, we have

$$\begin{aligned}a_0+\sum_{n=1}^m a_n\cos nx+\sum_{n=1}^m b_n\sin nx&=a_0+\sum_{n=1}^m a_n\,\tfrac{e^{inx}+e^{-inx}}{2}+\sum_{n=1}^m b_n\,\tfrac{e^{inx}-e^{-inx}}{2i}\\ &=a_0+\sum_{n=1}^m\big(\tfrac{a_n}{2}-i\,\tfrac{b_n}{2}\big)e^{inx}+\sum_{n=1}^m\big(\tfrac{a_n}{2}+i\,\tfrac{b_n}{2}\big)e^{-inx}=\sum_{n=-m}^m c_ne^{inx}\end{aligned}$$

where

$$c_0=a_0\ ,\ c_n=\tfrac12(a_n-ib_n)\ ,\ c_{-n}=\overline{c_n}=\tfrac12(a_n+ib_n)$$

for $n>0$. On the other hand, when $f\in\mathcal{R}(T)$ with say $a_n=a_n(f)$ and $b_n=b_n(f)$, if we let $c_0=a_0$ and $c_n=\frac12(a_n-i\,b_n)$ and $c_{-n}=\overline{c_n}=\frac12(a_n+i\,b_n)$, then

$$\begin{aligned}c_0&=a_0=\tfrac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx\\ c_n&=\tfrac12(a_n-i\,b_n)=\tfrac{1}{2\pi}\left(\int_{-\pi}^{\pi}f(x)\cos nx\;dx-i\int_{-\pi}^{\pi}f(x)\sin nx\;dx\right)\\ &=\tfrac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\big(\cos nx-i\,\sin nx\big)\,dx=\tfrac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-i\,nx}\,dx\end{aligned}$$

and similarly $c_{-n}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{inx}\,dx$. These formulas inspire the following definitions.

8.27 Definition: A (complex-valued) **trigonometric polynomial** is a function $p:\mathbb{R}\to\mathbb{C}$ of the form

$$p(x)=\sum_{n=-m}^m c_ne^{inx}$$

for some $c_n\in\mathbb{C}$, and we say that $p(x)$ is of degree m when either $c_m\neq0$ or $c_{-m}\neq0$. A (complex-valued) **trigonometric series** is a series of the form

$$s(x)=\sum_{n=-\infty}^{\infty}c_ne^{inx}$$

which denotes the sequence of partial sums $s_m(x)=\sum\limits_{n=-m}^m c_ne^{inx}$.

When $f:[-\pi,\pi]\to\mathbb{C}$ is Riemann integrable (or when $f:\mathbb{R}\to\mathbb{C}$ is 2π-periodic and its restriction to $[-\pi,\pi]$ is Riemann integrable), we define the (complex-valued) **Fourier coefficients** of f to be the complex numbers

$$c_n(f)=\tfrac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}\,dx,$$

we define the m^{th} **Fourier polynomial** of f to be

$$s_m(f)(x)=\sum_{n=-m}^m c_n(f)e^{inx}\,,$$

and we define the **Fourier series** of f to be the sequence of m^{th} Fourier polynomials.

The Dirichlet Kernel

8.28 Note: When $f:[-\pi,\pi]\to\mathbb{C}$ is Riemann integrable $\big($or $f:\mathbb{R}\to\mathbb{C}$ is 2π-periodic and its restriction to $[-\pi,\pi]$ is Riemann integrable$\big)$, we have

$$\begin{aligned} s_m(f)(x) &= \sum_{n=-m}^{m} c_n(f)\,e^{inx} = \sum_{n=-m}^{m}\left(\tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt\right)e^{inx} \\ &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{n=-m}^{m} e^{in(x-t)}\,dt = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_m(x-t)\,dt \end{aligned}$$

where $D_m(u)=\sum\limits_{n=-m}^{m} e^{inu}$. When $e^{iu}\neq 1$, that is when $u\neq 2\pi\,k$, we have

$$\begin{aligned} D_m(u) &= \sum_{n=-m}^{m} e^{inu} = e^{-imu}\,\frac{e^{i(2m+1)u}-1}{e^{iu}-1} = \frac{e^{i(m+1)u}-e^{-imu}}{e^{iu}-1}\cdot\frac{e^{-iu/2}}{e^{-iu/2}} \\ &= \frac{e^{i(2m+1)u/2}-e^{i(2m+1)u/2}}{e^{iu/2}-e^{-iu/2}} = \frac{\sin\frac{(2m+1)u}{2}}{\sin\frac{u}{2}}\,. \end{aligned}$$

8.29 Definition: The m^{th} **Dirichlet kernel** is the 2π-periodic function $D_m(f):\mathbb{R}\to\mathbb{R}$ given by

$$D_m(u)=\sum_{n=-m}^{m} e^{inu} = \begin{cases} \dfrac{\sin\frac{(2m+1)u}{2}}{\sin\frac{u}{2}} & \text{if } u\neq 2\pi k, k\in\mathbb{Z} \\ 2m+1 & \text{if } k=2\pi k, k\in\mathbb{Z} \end{cases}$$

8.30 Note: We have

$$\int_{-\pi}^{\pi} D_m(u)\,du = \int_{-\pi}^{\pi}\sum_{n=-m}^{m} e^{inu}\,du = \int_{-\pi}^{\pi} 1+\sum_{n=1}^{m} 2\cos(nu)\;du = 2\pi$$

and

$$\begin{aligned} \int_{-\pi}^{\pi}\big|D_m(u)\big|\,du &= \int_{-\pi}^{\pi}\left|\frac{\sin\frac{(2m+1)u}{2}}{\sin\frac{u}{2}}\right|du = 2\int_0^{\pi}\left|\frac{\sin\frac{(2m+1)u}{2}}{\sin\frac{u}{2}}\right|du \\ &\geq 2\int_{u=0}^{\pi}\frac{\left|\sin\frac{(2m+1)u}{2}\right|}{\frac{u}{2}}\,du = 2\int_{t=0}^{(m+\frac12)\pi}\frac{|\sin t|}{\frac{t}{2m+1}}\cdot\tfrac{2}{2m+1}\,dt \\ &\geq 4\sum_{n=1}^{m}\int_{(n-1)\pi}^{n\pi}\frac{|\sin t|}{t}\,dt \geq 4\sum_{n=1}^{m}\int_{(n-1)\pi}^{n\pi}\frac{|\sin t|}{n\pi}\,dt \\ &= \tfrac{8}{\pi}\sum_{n=1}^{m}\tfrac{1}{n} \geq \tfrac{8}{\pi}\int_{x=1}^{m+1}\tfrac{1}{x}\,dx = \tfrac{8}{\pi}\ln(m+1)\geq\tfrac{8}{\pi}\,\ln m\,. \end{aligned}$$

8.31 Theorem: *(Pointwise Divergence)*
(1) There exists a continuous function $f\in\mathcal{C}(T)$ whose Fourier series diverges at 0.
(2) For every sequence $(a_n)_{n\geq 1}$ of distinct points in $[-\pi,\pi]$, there exists $f\in\mathcal{C}(T)$ whose Fourier series diverges at every point a_k.
(3) For every sequence $(a_n)_{n\geq 1}$ of distinct points in $[-\pi,\pi]$, the set of functions $f\in\mathcal{C}(T)$ whose Fourier series diverges at every point a_k is dense in $\big(\mathcal{C}(T),\|\ \|_\infty\big)$.

Proof: The proof of this theorem is a bit beyond the level of this course, but we provide a proof in Appendix 2 in case anyone is interested. The proof uses the Dirichlet kernel.

The Fejér Kernel and Convergence of the Cesàro Means)

8.32 Theorem: *(Cesàro Summation) Let* $a_n \in \mathbb{C}$ *for* $n \geq 0$*, let* $s_m = \sum\limits_{n=0}^{m} a_n$ *and let*

$$\sigma_\ell = \tfrac{1}{\ell+1} \sum_{m=0}^{\ell} s_m \,.$$

If the sequence $(s_m)_{m\geq 0}$ *converges then so does the sequence* $(\sigma_\ell)_{\ell\geq 0}$ *and, in this case*

$$\lim_{\ell\to\infty} \sigma_\ell = \lim_{m\to\infty} s_m.$$

Proof: The proof is left as an exercise.

8.33 Definition: When $f : [-\pi, \pi] \to \mathbb{C}$ is Riemann integrable $\big($or when $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic and its restriction to $[-\pi, \pi]$ is Riemann integrable$\big)$, we define the ℓ^{th} **Cesàro mean** of the Fourier series of f to be the 2π-periodic function $\sigma_\ell(f) : \mathbb{R} \to \mathbb{C}$ given by

$$\sigma_\ell(f) = \tfrac{1}{\ell+1} \sum_{m=0}^{\ell} s_m(f).$$

8.34 Note: When $f : [-\pi, \pi] \to \mathbb{C}$ is Riemann integrable $\big($or $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic and its restriction to $[-\pi, \pi]$ is Riemann integrable$\big)$, we have

$$\begin{aligned}
\sigma_\ell(f)(x) &= \frac{1}{\ell+1} \sum_{m=0}^{\ell} s_m(f)(x) = \frac{1}{\ell+1} \sum_{m=0}^{\ell} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_m(x-t)\,dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, \tfrac{1}{\ell+1} \sum_{m=0}^{\ell} D_m(x-t)\,dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) K_\ell(x-t)\,dt
\end{aligned}$$

where $K_\ell(u) = \tfrac{1}{\ell+1} \sum\limits_{m=0}^{\ell} D_m(u)$. When $u \neq 2\pi k$ with $k \in \mathbb{Z}$ we have

$$\begin{aligned}
K_\ell(u) &= \frac{1}{\ell+1} \sum_{m=0}^{\ell} D_m(u) = \frac{1}{\ell+1} \sum_{m=0}^{\ell} \frac{\sin \frac{(2m+1)u}{2}}{\sin \frac{u}{2}} \\
&= \frac{1}{(\ell+1)\sin\frac{u}{2}} \cdot \text{Im}\Big(\sum_{m=0}^{\ell} e^{i(2m+1)u/2} \Big) = \frac{1}{(\ell+1)\sin\frac{u}{2}} \cdot \text{Im}\Big(e^{iu/2} \sum_{m=0}^{\ell} e^{imu} \Big) \\
&= \frac{1}{(\ell+1)\sin\frac{u}{2}} \cdot \text{Im}\Big(e^{iu/2}\, \frac{e^{i(\ell+1)u} - 1}{e^{iu} - 1} \Big) = \frac{1}{(\ell+1)\sin\frac{u}{2}} \cdot \text{Im}\Big(\frac{e^{i(\ell+1)u} - 1}{e^{iu/2} - e^{\frac{-iu}{2}}} \Big) \\
&= \frac{1}{(\ell+1)\sin\frac{u}{2}} \cdot \text{Im}\Big(\frac{e^{i(\ell+1)u/2} - e^{-i(\ell+1)u/2}}{e^{iu/2} - e^{-iu/2}} \cdot e^{i(\ell+1)u/2} \Big) \\
&= \frac{1}{(\ell+1)\sin\frac{u}{2}} \cdot \frac{\sin\frac{(\ell+1)u}{2}}{\sin\frac{u}{2}} \cdot \sin\frac{(\ell+1)u}{2} = \frac{\sin^2\frac{(\ell+1)u}{2}}{(\ell+1)\,\sin^2\frac{u}{2}}\,.
\end{aligned}$$

8.35 Definition: The above function $K_\ell : \mathbb{R} \to \mathbb{R}$ is called the ℓ^{th} **Féjer kernel**.

8.36 Theorem: *We have*

(1) For $0 < t \leq \pi$ *we have* $0 \leq K_\ell(t) \leq \frac{\pi^2}{(\ell+1)\,t^2}$.

(2) $\displaystyle\int_{-\pi}^{\pi} K_\ell(t)\,dt = 2\int_0^{\pi} K_\ell(t)\,dt = 2\pi.$

(3) $\displaystyle\int_{\pi}^{\pi} f(t)K_\ell(x-t)\,dt = \int_{-\pi}^{\pi} f(x+t)K_\ell(t)\,dt = \int_{-\pi}^{\pi} f(x-t)K_\ell(t)\,dt.$

Proof: The proof is left as an exercise. These results will needed for Theorem 8.37.

8.37 Theorem: *(Fejér's Theorem on Convergence of the Cesàro Means) Let* $f:\mathbb{R}\to\mathbb{C}$ *be a* 2π*-periodic function whose restriction to* $[-\pi,\pi]$ *is Riemann integrable.*

(1) If $a\in\mathbb{R}$ *and the one-sided limits* $f(a^-)=\lim\limits_{x\to a^-}f(x)$ *and* $f(a^+)=\lim\limits_{x\to a^+}f(x)$ *both exist in* $\mathbb{R}$*, then*

$$\lim_{\ell\to\infty}\sigma_\ell(f)(a)=\frac{f(a^-)+f(a^+)}{2}\,.$$

(2) If $a,b\in\mathbb{R}$ *with* $a\le b$ *and* f *is continuous in* $[a,b]$ *then* $\sigma_\ell\to f$ *uniformly on* $[a,b]$.

Proof: By Part 3 of Theorem 8.36, we have

$$\sigma_\ell(f)(a)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)K_\ell(a-t)\,dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(a+t)+f(a-t)}{2}\,K_\ell(t)\,dt$$

and by Part 2 of Theorem 8.36 we have

$$\frac{f(a^+)+f(a^-)}{2}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(a^+)+f(a^-)}{2}\,K_\ell(t)\,dt$$

and so

$$\begin{aligned}\left|\sigma_\ell(f)(a)-\frac{f(a^+)+f(a^-)}{2}\right|&=\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{f(a+t)+f(a-t)}{2}-\frac{f(a^+)+f(a^-)}{2}\right)K_\ell(t)\;dt\right|\\&=\left|\frac{1}{2\pi}\int_0^{\pi}\Big((f(a+t)-f(a^+))+(f(a-t)-f(a^-))\Big)K_\ell(t)\;dt\right|\\&\le\frac{1}{2\pi}\int_0^{\pi}\Big(\big|f(a+t)-f(a^+)\big|+\big|f(a-t)-f(a^-)\big|\Big)\,K_\ell(t)\;dt\\&=I_\delta+J_\delta,\end{aligned}$$

for any $0<\delta\le\pi$, where

$$\begin{aligned}I_\delta&=\frac{1}{2\pi}\int_0^{\delta}\Big(\big|f(a+t)-f(a^+)\big|+\big|f(a-t)-f(a^-)\big|\Big)\,K_\ell(t)\;dt\\J_\delta&=\frac{1}{2\pi}\int_\delta^{\pi}\Big(\big|f(a+t)-f(a^+)\big|+\big|f(a-t)-f(a^-)\big|\Big)\,K_\ell(t)\;dt\,.\end{aligned}$$

Let $\epsilon>0$. Choose $\delta>0$ so that for all $0<t<\delta$ we have $\big|f(x+t)-f(a^+)\big|<\frac{\epsilon}{2}$ and $\big|f(x-t)-f(a^-)\big|<\frac{\epsilon}{2}$. Then, by Part 2 of Theorem 8.36,

$$I_\delta\le\frac{1}{2\pi}\int_0^{\pi}\epsilon\cdot K_\ell(t)\,dt\le\tfrac{\epsilon}{2}.$$

By Part 1 of Theorem 8.36, for $\delta\le t\le\pi$ we have $K_\ell(t)\le\frac{\pi^2}{(\ell+1)\,\delta^2}$ so for $\ell+1\ge\frac{M}{\epsilon}$ where $M=\pi\big(\|f\|_1+\pi|f(a^+)+f(a^-)|\big)/\delta^2$ we have

$$\begin{aligned}J_\delta&\le\frac{1}{2\pi}\int_\delta^{\pi}\Big(|f(a+t)|+|f(a-t|+|f(a^+)+f(a^-)|\Big)\,\tfrac{\pi^2}{(\ell+1)\,\delta^2}\;dt\\&\le\tfrac{1}{2\pi}\cdot\tfrac{\pi^2}{(\ell+1)\,\delta^2}\big(\|f\|_1+\pi\big|f(a^+)+f(a^-)\big|=\tfrac{M}{2(\ell+1)}\le\tfrac{\epsilon}{2}.\end{aligned}$$

This proves Part (1), and Part (2) can be proven using the same method noting that the estimates can be made uniformly.

8.38 Corollary: *Let $f \in \mathcal{R}(T)$, extend f to a 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$, and let $a \in \mathbb{R}$. If $f(a^+)$, $f(a^-)$ and $\lim_{m\to\infty} s_m(f)(a)$ all exist in $\mathbb{R}$ then*

$$\lim_{m\to\infty} s_m(f)(a) = \frac{f(a^+) + f(a^-)}{2}.$$

8.39 Corollary: *Let $f \in \mathcal{R}(T)$, extend f to a 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$, and let $a \in \mathbb{R}$. If f is continuous at a and $\lim_{m\to\infty} s_m(f)(a)$ exists in $\mathbb{R}$ then $\lim_{m\to\infty} s_m(f)(a) = f(a)$.*

8.40 Example: In Example 8.8, we considered the 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{\pi}{2} + x$ when $-\pi \le x \le 0$ and by $f(x) = \frac{\pi}{2} - x$ when $0 \le x \le \pi$, and we found that its Fourier series is given by

$$s(f)(x) = \sum_{k=0}^{\infty} \tfrac{4}{\pi(2k+1)^2} \cos((2k+1)x) = \tfrac{4}{\pi}\left(\tfrac{1}{1^2}\cos x + \tfrac{1}{3^2}\cos 3x + \tfrac{1}{5^2}\cos 5x + \cdots\right).$$

Since $\left|\frac{4}{\pi(2k+1)^2}(\cos(2k+1)x)\right| \le \frac{4}{\pi(2k+1)^2}$ and $\sum_{k=1}^{\infty} \frac{4}{\pi(2k+1)^2}$ converges (say by the Integral Test), it follows that the Fourier series of f converges uniformly on $\mathbb{R}$ to some continuous function $g : \mathbb{R} \to \mathbb{R}$ (but it is not immediately obvious that $g = f$). Since $s_m(f) \to g$ uniformly on $\mathbb{R}$, we also have $s_m(f) \to g$ pointwise on $\mathbb{R}$, that is $\lim_{m\to\infty} s_m(f)(x) = g(x)$ for all $x \in \mathbb{R}$. By Theorem 8.32 (whose proof was left as an exercise) when a series converges, so does the sequence of Cesàro means, and they have the same sum, so we have $\lim_{\ell\to\infty} \sigma_\ell(f)(x) = \lim_{m\to\infty} s_m(f)(x) = g(x)$ for all $x \in \mathbb{R}$. By Fejer's Theorem (whose proof was not easy), since f is continuous, we have $\lim_{\ell\to\infty} \sigma_\ell(f)(x) = f(x)$ for all $x \in \mathbb{R}$. Thus we have $s(f)(x) = \lim_{m\to\infty} s_m(f)(x) = \lim_{\ell\to\infty} \sigma_\ell(f)(x) = f(x)$ for all $x \in \mathbb{R}$. In particular, we have $f(0) = s(f)(0)$, that is $\frac{\pi}{2} = \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^3} + \frac{1}{5^2} + \cdots\right)$, and hence $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$.

8.41 Example: Use Parseval's Identity, together with the result of Example 8.8, to prove that $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}$ and use this result to calculate $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Solution: From our work in Example 8.8, we know that $a_n(f) = 0$ when n is even, and $a_n(f) = \frac{4}{\pi n^2}$ when n is odd, and $b_n(f) = 0$ for all n, so by Parceval's Theorem

$$\|f\|_2{}^2 = 2\pi\, a_0(f)^2 + \pi \sum_{n=1}^{\infty} \left(a_n(f)^2 + b_n(f)^2\right) = \pi \sum_{n \text{ odd}} \left(\tfrac{4}{\pi n^2}\right)^2 = \sum_{k=0}^{\infty} \tfrac{16}{\pi(2k+1)^4}.$$

On the other hand, we have

$$\|f\|_2{}^2 = \int_{-\pi}^{\pi} |f(x)|^2\, dx = 2\int_0^{\pi} \left(\tfrac{\pi}{2} - x\right)^2 dx = 2\int_0^{\pi} \tfrac{\pi^2}{4} - \pi x + x^2\, dx = 2\left(\tfrac{\pi^3}{.4} - \tfrac{\pi^3}{2} + \tfrac{\pi^3}{3}\right) = \tfrac{\pi^3}{6}$$

Thus we have $\sum_{k=0} \frac{16}{\pi(2k+1)^4} = \frac{\pi^3}{6}$ and hence $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}$.

Letting $S = \sum_{n=1}^{\infty} \frac{1}{n^4}$, we have

$$S = \sum_{n \text{ even}} \tfrac{1}{n^4} + \sum_{n \text{ odd}} \tfrac{1}{n^4} = \sum_{k=1}^{\infty} \tfrac{1}{(2k)^4} + \sum_{k=0}^{\infty} \tfrac{1}{(2k+1)^4} = \tfrac{1}{16}S + \tfrac{\pi^4}{96}$$

so that $\frac{15}{16}S = \frac{\pi^4}{96}$. Thus $S = \frac{16}{15} \cdot \frac{\pi^4}{96} = \frac{\pi^4}{90}$, that is $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

Appendix 1. Applications of Fourier Series to Differential Equations

8.42 Example: (Forced Damped Oscillations) Suppose an object of mass m is attached to a spring of spring-constant k and vibrates in a fluid of damping-constant c and let $x = x(t)$ be the displacement of the object from its equilibrium position at time t. Suppose, in addition, that the object is acted on by an external force $f(t)$. The total force $F(t)$ acting on the object consists of the force exerted by the spring, which is equal to $-kx(t)$, the resistive force exerted by the fluid, which is equal to $-cx'(t)$, and the external driving force, which is equal to $f(t)$. By Newton's Second Law of motion we have $F(t) = mx''(t)$ and so $x(t)$ satisfies the differential equation (the DE)

$$mx''(t) + cx'(t) + kx(t) = f(t).$$

8.43 Example: Use Fourier series to solve the above DE with $m = 1$, $c = 2$ and $k = 5$, where $f(t)$ is the function from Example 8.8,

Solution: We need to solve the DE

$$x''(t) + 2x'(t) + 5x(t) = f(t).$$

To solve the associated homogeneous DE $x'' + 2x' + 5x = 0$, we look for a solution of the form $x = x(t) = e^{rt}$. Putting $x = e^{rt}$, $x' = re^{rt}$ and $x'' = r^2e^{rt}$ into the homogeneous DE gives $(r^2+2r+5)e^{rt} = 0$ hence $r = -1\pm 2i$. This gives us the two complex-valued solutions $x(t) = e^{(-1\pm 2i)t} = e^{-t}(\cos 2t \pm i \sin 2t)$. By taking suitable linear combinations of these two complex-valued solutions we obtain the two real-valued solutions $x_1(t) = e^{-t}\cos 2t$ and $x_2(t) = e^{-t}\sin 2t$. The general solution to the DE $x'' + 2x' + 5x = 0$ is given by

$$x(t) = Ae^{-t}\cos 2t + Be^{-t}\sin 2t \quad \text{, where } A, B \in \mathbb{R}.$$

For each $n \in \mathbb{Z}^+$, to find a particular solution to the DE $x''+2x'+5x = \cos nt$, we look for a solution of the form $x = x(t) = A_n \cos nt + B_n \sin nt$. Putting $x = A_n \cos nt + B_n \sin nt$, $x' = -nA_n \sin nx + nB_n \cos nt$ and $x'' = -n^2A_n \cos nt - n^2B_n \sin nt$ into $x'' + 2x' + 5 = \cos nt$ gives $\big(-n^2A_n + 2nB_n + 5A_n\big)\cos nt + \big(-n^2B_n - 2nA_n + 5B_n\big)\sin nt = \cos nt$ for all $t \in \mathbb{R}$ and so we must have $(5-n^2)A_n + 2nB_n = 1$ and $(5-n^2)B_n - 2nA_n = 0$. We solve these two equations to get $A_n = \frac{5-n^2}{n^4-6n^2+25}$ and $B_n = \frac{2n}{n^4-6n^2+25}$ and so one solution to the DE $x'' + 2x' + 5 = \cos nt$ is given by

$$x(t) = A_n \cos nt + B_n \sin nx \text{ , where } A_n = \tfrac{5-n^2}{n^4-6n^2+25} \text{ and } B_n = \tfrac{2n}{n^4-6n^2+25}.$$

Since $f(t) = \sum\limits_{n \text{ odd}} \frac{4}{\pi n^2}\cos nt$, one particular solution, called the **steady state solution**, to the original DE $x'' + 2x' + 5x = f(t)$ is given by

$$x(t) = \sum_{n \text{ odd}} \frac{4}{\pi n^2}\big(A_n \cos nt + B_n \sin nt\big)$$

and the general solution is

$$x(t) = Ae^{-t}\cos 2t + Be^{-t}\sin 2t + \sum_{n \text{ odd}} \frac{4}{\pi n^2}\big(A_n \cos nt + B_n \sin nt\big) \quad \text{, where } A, B \in \mathbb{R}.$$

8.44 Example: (The One-Dimensional Wave Equation) An elastic string is stretched to length π and is fixed at its two endpoints along the x-axis at $x=0$ and $x=\pi$. The string is displaced so that it follows the curve $u=f(x)$ with $f(0)=0$ and $f(\pi)=0$, then at time $t=0$ the string is released and allowed to vibrate. The problem is to determine the strings shape $u=u(x,t)$ at all points $0\le x\le\pi$ and all times $t\ge 0$.

To formulate a differential equation (or DE) which models the situation, we consider a segment of string, at time t, between the points $p_1=(x_1,u(x_1,t))$ and $p_2=(x_2,u(x_2,t))$ where the difference $dx=x_2-x_1$ is small. The slope of the curve $u=g(x)=u(x,t)$ at p_1 is $\frac{\partial u}{\partial x}(x_1,t)$ and the angle θ_1 from the horizontal is given by $\tan\theta_1=\frac{\partial u}{\partial x}(x_1,t)$. Similarly, the angle θ_2 at p_2 is given by $\tan\theta_2=\frac{\partial u}{\partial x}(x_2,t)$, and we have

$$\tan\theta_2-\tan\theta_1=\frac{\partial u}{\partial x_1}(x_1,t)-\frac{\partial u}{\partial x}(x_2,t)=\frac{\partial^2 u}{\partial x^2}\,dx.$$

Let T_1 be the magnitude of the force exerted on p_1 by the portion of the string which lies to the left of p_1, and let T_2 be the magnitude of the force exerted on p_2 by the portion of the string which lies to the right of p_2. Assuming that the segment of string moves only vertically (so the total horizontal component of the force is zero) we have $T_1\cos\theta_1=T_2\cos\theta_2$. Let

$$T=T_1\cos\theta_1=T_2\cos\theta_2$$

and note that T is a constant which we call the **tension** of the string. The total vertical component of the force is $F=T_2\sin\theta_2-T_1\sin\theta_1$ and by Newton's Second Law of motion, we have

$$T_2\sin\theta_2-T_2\sin\theta_1=m\frac{\partial^2 u}{\partial t^2}=\rho\,dx\frac{\partial^2 u}{\partial t^2}$$

where ρ is the linear **density** of the string, that is its mass per unit length. From the equations $\tan\theta_2-\tan\theta_1=\frac{\partial^2 u}{\partial x^2}\,dx$, $T_1\cos\theta_1=T_2\cos\theta_2$ and $T_2\sin\theta_2-T_1\sin\theta_2=\rho\,dx\,\frac{\partial^2 u}{\partial t^2}$ we obtain the **one-dimentional wave equation**

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}\text{ , where } c^2=\frac{T}{\rho}.$$

8.45 Example: Use Fourier series to solve the one-dimensional wave equation $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$ subject to the boundary conditions $u(0,t)=0$ and $u(0,\pi)=0$ for all $t\ge 0$ and to the initial conditions $u(x,0)=f(x)$ and $\frac{\partial u}{\partial t}(x,0)=g(x)$ for all $0\le x\le\pi$.

Solution: We use a method known as **separation of variables**. We look for a solution to the DE of the form $u(x,t)=y(x)s(t)$ which satisfies the given boundary conditions $0=u(0,t)=y(0)s(t)$ and $0=u(\pi,t)=y(\pi)s(t)$. If we had $y(x)=0$ for all x or $s(t)=0$ for all t then we would obtain the trivial solution $u(x,t)=0$ for all x,t, so let us assume this is not the case, so the boundary conditions become $y(0)=y(\pi)=0$. When $u(x,t)=y(x)s(t)$, the DE becomes $y(x)s''(t)=c^2y''(x)s(t)$ which we can write as $\frac{y''(x)}{y(x)}=\frac{1}{c^2}\frac{s''(t)}{s(t)}$. Since the function on the left is a function of x (and is constant in t) and the function on the right is a function of t (and is constant in x), in order for these two functions to be equal for all x,t they must both be constant, say

$$\frac{y''(x)}{y(x)}=k=\frac{1}{c^2}\frac{s''(t)}{s(t)}$$

where k is constant.

First we solve the DE $\frac{y''(x)}{y(x)} = k$ subject to the boundary conditions $y(0) = y(\pi) = 0$. If $k = 0$ then the DE becomes $y'' = 0$, which has solution $y = Cx + D$, and the boundary conditions give $C = D = 0$, so we obtain the trivial solution. If $k > 0$, say $k = n^2$ where $n > 0$, then the DE becomes $y'' - n^2 y = 0$, which has solution $y = Ce^{nx} + De^{-nx}$, and the boundary conditions give $C + D = 0$ and $Ce^{n\pi} + De^{-n\pi} = 0$ which imply that $C = D = 0$, so again we obtain the trivial solution. Suppose that $k < 0$, say $k = -n^2$ where $n > 0$. The DE becomes $y'' + n^2 y = 0$ which has solution $y = C\cos nx + D\sin nx$. The boundary condition $y(0) = 0$ gives $C = 0$ so that $y = D\sin nx$, and the boundary condition $y(\pi) = 0$ gives $D = 0$ or $\sin n\pi = 0$. If $D = 0$ we obtain the trivial solution and if $\sin n\pi = 0$ then we must have $n \in \mathbb{Z}$. Thus in order to obtain a nontrivial solution to the DE which satisfies the boundary conditions we must have $k = -n^2$ for some $n \in \mathbb{Z}^+$ and, in this case,

$$y(x) = D_n \sin nx \text{ , where } D_n \in \mathbb{R}.$$

When $k = -n^2$ with $n \in \mathbb{Z}^+$, and $y(x) = D_n \sin nx$, the DE $\frac{1}{c^2}\frac{s''(t)}{s(t)} = k.$ becomes $s''(t) + (cn)^2 s(t) = 0$, and the solution is $s(t) = A_n \cos(cn\,t) + B_n \sin(cn\,t)$. Thus, for each $n \in \mathbb{Z}^+$, and for all $A_n, B_n \in \mathbb{R}$, the function

$$u(x,t) = y(x)s(t) = \big(A_n \cos cnt + B_n \sin cnt\big)\sin nx$$

is a solution to the one-dimensional wave equation which satisfies the boundary conditions (we remark that it would be redundant to include the constants D_n as they could be amalgamated with the constants A_n and B_n).

In order to find a solution which satisfies the given initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$, we look for a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} \big(A_n \cos cnt + B_n \sin cnt\big)\sin nx\,.$$

In order to obtain $u(x,0) = f(x)$ we need $\sum_{n=1}^{\infty} A_n \sin nx = f(x)$ and so we choose A_n to be equal to the Fourier coefficients of the odd 2π-periodic function $F(x)$ with $F(x) = f(x)$ for $0 \le x \le \pi$, that is we choose

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\;dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\;dx\,.$$

Assuming that we can differentiate term-by-term, we have

$$\tfrac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty}\big(-cnA_n \sin cnt + cnB_n \cos nt\big)\sin nx.$$

In order to obtain $\frac{\partial u}{\partial t}(x,0) = g(x)$ we need $\sum_{n=1}^{\infty} cnB_n \sin nx = g(x)$ and so we choose B_n to be equal to the Fourier coefficients of the odd 2π-periodic function $G(x)$ with $G(x) = g(x)$ for $0 \le x \le \pi$, that is

$$B_n = \frac{2}{cn\pi}\int_0^{\pi} g(x)\sin nx\;dx\,.$$

Appendix 2. Continuous Functions with Divergent Fourier Series

8.46 Definition: Let X be a metric space and let $A \subseteq X$. Recall that A is said to be **dense** (in X) when for every nonempty open ball $B \subseteq X$ we have $B \cap A \neq \emptyset$, or equivalently when $\overline{A} = X$. We say A is **nowhere dense** (in X) when for every nonempty open ball $B \subseteq \mathbb{R}$ there exists a nonempty open ball $C \subseteq B$ with $C \cap A = \emptyset$, or equivalently when $\overline{A}^{\,0} = \emptyset$.

8.47 Note: When $A \subseteq B \subseteq X$, note that if A is dense in X then so is B and, on the other hand, if B is nowhere dense in X then so is A.

8.48 Note: When $A, B \subseteq X$ with $B = A^c = X \setminus A$, note that A is nowhere dense $\iff$ $\overline{A}^{\,0} = \emptyset \iff \overline{B^0} = X \iff$ the interior of B is dense.

8.49 Definition: Let $A \subseteq X$. We say that A is **first category** (or that A is **meagre**) when A is equal to a countable union of nowhere dense sets. We say that A is **second category** when it is not first category. We say that A **residual** when A^c is first category.

8.50 Example: If $(a_n)_{n \geq 1}$ is a sequence in $\mathbb{R}$, then the set $A = \{a_1, a_2, a_3, \cdots\}$ is first category because $A = \bigcup\limits_{k=1}^{\infty} \{a_k\}$. In particular $\mathbb{Q}$ is first category and $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$ is residual.

8.51 Note: If $A \subseteq X$ is first category then so is every subset of A.

8.52 Note: If $A_1, A_2, A_3, \cdots \subseteq X$ are are all first category then so is $\bigcup\limits_{k=1}^{\infty} A_k$.

8.53 Theorem: *(The Baire Category Theorem) Let X be a complete metric space.*
(1) Every first category set in X has an empty interior.
(2) Every residual set in X is dense.
(3) Every countable union of closed sets with empty interiors in X has an empty interior.
(4) Every countable intersection of dense open sets in X is dense.

Proof: Parts (1) and (2) are equivalent by taking complements, and Parts (3) and (4) are special cases of Parts (1) and (2), so it suffices to prove Part (1). We sketch a proof.

Let $A \subseteq X$ be first category, say $A = \bigcup\limits_{n=1}^{\infty} C_n$ where each C_n is nowhere dense. Suppose, for a contradiction, that A has nonempty interior, and choose an open ball $B_0 = B(a_0, r_0)$ with $0 < r_0 < 1$ such that $\overline{B}_0 \subseteq A$. Since each C_n is nowhere dense, we can chose a nested sequence of open balls $B_n = B(a_n, r_n)$ with $0 < r_n < \frac{1}{2^n}$ such that $\overline{B}_n \subseteq B_{n-1}$ and $\overline{B}_n \cap C_n = \emptyset$. Because $r_n \to 0$, it folows that the sequence $\{a_n\}$ is Cauchy. Because X is complete, it follows that $\{a_n\}$ converges in X, say $a = \lim\limits_{n \to \infty} a_n$. Note that $a \in \overline{B}_n$ for all n since $a_k \in \overline{B}_n$ for all $k \geq n$. Since $a \in \overline{B}_0$ and $\overline{B}_0 \subseteq A$ we have $a \in A$. But since $a \in \overline{B}_n$for al $n \geq 1$, and $\overline{B}_n \cap C_n = \emptyset$, we have $a \notin C_n$ for all $n \geq 1$ hence $a \notin \bigcup\limits_{n=1}^{\infty} C_n$, that is $a \notin A$.

8.54 Example: Recall that $\mathbb{Q}$ is first category and $\mathbb{Q}^c$ is residual. The Baire Category Theorem shows that $\mathbb{Q}^c$ cannot be first category because if $\mathbb{Q}$ and $\mathbb{Q}^c$ were both first category then $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ would also be first category, but this is not possible since $\mathbb{R}$ does not have empty interior.

8.55 Exercise: For each $n \in \mathbb{Z}^+$, let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that for all $x \in \mathbb{R}$ there exists $n \in \mathbb{Z}^+$ such that $f_n(x) \in \mathbb{Q}$. Prove that there exists $n \in \mathbb{Z}^+$ such that f_n is constant in some nondegenerate interval.

8.56 Definition: When X and Y are normed linear spaces over $F = \mathbb{R}$ or $\mathbb{C}$, a linear map $L : X \to Y$ is also called a **linear operator**, and a linear map $L : X \to F$ is also called a **linear functional** on X.

8.57 Definition: Let X and Y be normed linear spaces and let $L : X \to Y$ be a linear operator. The **operator norm** of L is given by

$$\|L\| = \sup\left\{ \|Lx\| \,\middle|\, x \in X \text{ with } \|x\| \le 1 \right\}$$

and we say that L is **bounded** when $\|L\| < \infty$. Since $Lx = \|x\|\, L\big(\frac{x}{\|x\|}\big)$ for all $0 \ne x \in X$, it follows that

$$\|Lx\| \le \|L\|\,\|x\| \text{ for all } x \in X.$$

Recall (from Theorem 5.35) that L is bounded if and only if L is continuous. Also recall that a **Banach space** is a complete normed linear space.

8.58 Theorem: *(The Uniform Boundedness Principle) Let X be a Banach space and let Y be a normed linear space. Let S be a set of bounded linear operators $L : X \to Y$. Suppose that for every $x \in X$ there exists $m_x \ge 0$ such that $\|Lx\| \le m_x$ for all $L \in S$. Then there exists $m \ge 0$ such that $\|L\| \le m$ for all $L \in S$.*

Proof: For each $n \in \mathbb{Z}^+$, let $A_n = \left\{ x \in X \,\middle|\, \|Lx\| \le n \text{ for all } L \in S \right\}$. Note that A_n is closed because the sets $\left\{ x \in X \,\middle|\, \|Lx\| \le n \right\}$ are closed for each $L \in S$, and A_n is the intersection of these sets. By the hypothesis of the theorem, we have $X = \bigcup_{n=1}^{\infty} A_n$. By the Baire Category Theorem (since X is complete), the sets A_n cannot all be nowhere dense. Choose $n \in \mathbb{Z}^+$ so that A_n is not nowhere dense. Chose $a \in A_n$ and $r > 0$ so that $\overline{B}(a, r) \subseteq A_n$. For all $x \in X$, if $x \in B(a, r)$ then $x \in A_n$ so we have $\|L(x)\| \le n$ for all $L \in S$. If $\|x\| < r$ then $x + a \in B(a, r)$ and $a \in B(a, r)$ and so

$$\|L(x)\| = \|L(x + a) - L(a)\| \le \|L(x + a)\| + \|L(a)\| \le 2n \;\text{ for all } L \in S.$$

For all $L \in S$ and $x \in X$, if $\|x\| \le 1$ then $\|rx\| \le r$ and so $\|L(x)\| = \frac{1}{r}\,\|L(rx)\| \le \frac{2n}{r}$. Thus we have $\|L\| \le \frac{2n}{r}$ for all $L \in S$.

8.59 Theorem: *(Condensation of Singularities) Let X be a Banach space and let Y be a normed linear space. For each $m, n \in \mathbb{Z}^+$, let $L_{m,n} : X \to Y$ be a bounded linear operator. Suppose that for each $m \in \mathbb{Z}^+$ there exists $x_m \in X$ such that $\limsup\limits_{n\to\infty} \|L_{m,n}(x_m)\| = \infty$. Then the set $E = \left\{ x \in X \,\middle|\, \limsup\limits_{n\to\infty} \|L_{m,n}(x)\| = \infty \text{ for all } m \in \mathbb{Z}^+ \right\}$ is dense in X.*

Proof: Fix $m \in \mathbb{Z}^+$. For each $\ell \in \mathbb{Z}^+$, let $A_\ell = \left\{ x \in X \,\middle|\, \|L_{n,m}(x)\| \le \ell \text{ for all } n \in \mathbb{Z}^+ \right\}$ and note that each set A_ℓ is closed. As in the proof of the Uniform Boundedness Principle, if one of the sets A_ℓ was not nowhere dense then we could choose $m \ge 0$ such that $\|L_{m,n}\| \le m$ for all $n \in \mathbb{Z}^+$. But then for all $x \in X$ we would have $\|L_{m,n}(x)\| \le m\|x\|$ for all n so that $\limsup\limits_{n\to\infty} \|L_{m,n}(x)\| \le m\|x\|$, contradicting the hypothesis of the theorem. Thus all of the sets A_ℓ must be nowhere dense. Let $B_m = \bigcup\limits_{\ell=1}^{\infty} A_\ell = \left\{ x \in X \,\middle|\, \limsup\limits_{n\to\infty} \|L_{m,n}(x)\| < \infty \right\}$ and let $C = \bigcup\limits_{m=1}^{\infty} B_m = \left\{ x \in X \,\middle|\, \limsup\limits_{n\to\infty} \|L_{m,n}(x)\| < \infty \text{ for some } m \in \mathbb{Z}^+ \right\}$, and note that $E = X \setminus C$. Then C is a countable union of closed nowhere dense sets, so E is a countable intersection of open dense sets. By the Baire Category Theorem, E is dense.

8.60 Theorem: *(Pointwise Divergence)*
(1) There exists a function $f \in \mathcal{C}(T)$ whose Fourier series diverges at 0.
(2) For every sequence $(a_n)_{n\geq 1}$ of distinct points in $[-\pi,\pi]$, the set of functions $f \in \mathcal{C}(T)$ whose Fourier series diverges at every point a_k is dense in $\big(\mathcal{C}(T), \|\ \ \|_\infty\big)$.

Proof: First we fix $x=0$. For $m \in \mathbb{Z}^+$, define $F_m : C(T) \to \mathbb{R}$ by

$$F_m(f) = s_m(f)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, D_m(t)\, dt.$$

Note that

$$\big|F_m(f)\big| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|f(t)\big|\,\big|D_m(t)\big|\, dt \leq \frac{1}{2\pi}\,\|f\|_\infty \int_{-\pi}^{\pi} \big|D_m(t)\big|\, dt$$

so we have

$$\|F_m\| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|D_m(t)\big|\, dt\,.$$

We claim that in fact $\|F_m\| = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \big|D_m(t)\big|\, dt$. Fix m and define

$$s(t) = \begin{cases} \ \ 1 \text{ if } D_m(t) \geq 0, \\ -1 \text{ if } D_m(t) < 0. \end{cases}$$

Construct continuous piecewise linear functions $\ell_n : [-\pi,\pi] \to \mathbb{R}$, as we did in the proof of Theorem 8.20, such that $\ell_n \to s$ in $\big(\mathcal{R}(T), \|\ \ \|_1\big)$ so that we have

$$F_m(\ell_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ell_n(t) D_m(t)\, dt \longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} s(t) D_m(t)\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|D_m(t)\big|\, dt.$$

It follows that $\|F_m\| = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \big|D_m(t)\big|\, dt$, as claimed. By Note 8.30, $\|F_m\| \geq \frac{8}{\pi}\ln m$, so the set of linear operators $S = \big\{F_m \big| m \in \mathbb{Z}^+\big\}$ is not uniformly bounded. By the Uniform Boundedness Principle, applied to the set S, there exists a function $f \in C(T)$ such that for all $M > 0$ we have $\big\|F_m(f)\big\| > M$, that is $\big|s_m(f)(0)\big| > M$, for some $m \in \mathbb{Z}^+$. For this function $f \in C(T)$, the Fourier series for f diverges at 0 because $\limsup\limits_{m\to\infty}\big|s_m(f)(0)\big| = \infty$. This completes the proof of Part 1.

Let $(a_n)_{n\geq 1}$ be a sequence in $[-\pi,\pi]$. For each $n \in \mathbb{Z}^+$ let $f_n(x) = f(x - a_n)$ and note that $\limsup\limits_{m\to\infty}\big|s_m(f_n)(a_n)\big| = \infty$. For each $n, m \in \mathbb{Z}^+$, define $L_{n,m} : C(T) \to \mathbb{R}$ by $L_{n,m}(f) = s_m(f)(a_n)$. By Condensation of Singularities, the set

$$E = \Big\{ f \in C(T) \,\Big|\, \limsup_{m\to\infty} \big\|L_{n,m}(f)\big\| = \infty \text{ for all } n \in \mathbb{Z}^+ \Big\}$$

is dense in the Banach space $C(T)$. For each $f \in E$, we have $\limsup\limits_{m\geq 0}\big|s_m(f)(a_n)\big| = \infty$ for every $n \in \mathbb{Z}^+$, so the Fourier series for f diverges at every point a_n.