Chapter 5. Limits and Continuity in Metric Spaces

Limits of Sequences in Metric Spaces

5.1 Definition: Let $(x_n)_{n\geq p}$ be a sequence in a metric space X. We say that the sequence $(x_n)_{n\geq p}$ is **bounded** when the set $\{x_n\}_{n\geq p}$ is bounded, that is when there exists $a\in X$ and $r>0$ such that $x_n\in B(a,r)$ for all indices $n\geq p$.

For $a\in X$, we say that the sequence $(x_n)_{n\geq p}$ **converges** to a (or that the **limit** of x_n is equal to a) and we write $\lim\limits_{n\to\infty} x_n=a$ (or we write $x_n\to a$) when

$$\forall\epsilon>0\ \exists m\in\mathbb{Z}_{\geq p}\ \forall n\in\mathbb{Z}_{\geq p}\big(n\geq m\implies d(x_n,a)<\epsilon\big).$$

We say that the sequence $(x_n)_{n\geq p}$ **converges** (in X) when it converges to some point $a\in X$, and otherwise we say that $(x_n)_{n\geq p}$ **diverges** (in X).

5.2 Theorem: *(Basic Properties of Limits of Sequences) Let $(x_n)_{n\geq p}$ be a sequence in a metric space X, and let $a\in X$.*

(1) If $(x_n)_{n\geq p}$ converges then its limit is unique.
(2) If $q\geq p$ and $y_n=x_n$ for all $n\geq q$, then $(x_n)_{n\geq p}$ converges if and only if $(y_n)_{n\geq q}$ converges and, in this case, $\lim\limits_{n\to\infty} y_n=\lim\limits_{n\to\infty} x_n$.
(3) If $(x_{n_k})_{k\geq q}$ is a subsequence of $(x_n)_{n\geq p}$, and $\lim\limits_{n\to\infty} x_n=a$, then $\lim\limits_{k\to\infty} x_{n_k}=a$.
(4) If $(x_n)_{n\geq p}$ converges then it is bounded.
(5) We have $\lim\limits_{n\to\infty} x_n=a$ in X if and only if $\lim\limits_{n\to\infty} d(x_n,a)=0$ in $\mathbb{R}$.

Proof: We prove Parts 1, 4 and 5 and leave the proofs of the other parts as an exercise. To prove Part 1, suppose that $x_n\to a$ in X and $x_n\to b$ in X. We need to show that $a=b$. Suppose, for a contradiction, that $a\neq b$, and note that $d(a,b)>0$. Since $x_n\to a$ and $x_n\to b$, we can choose $m\in\mathbb{Z}_{\geq p}$ such that when $n\geq m$ we have $d(x_n,a)<\frac{d(a,b)}{2}$, and $d(x_n,b)<\frac{d(a,b)}{2}$. Then we have $d(a,b)\leq d(a,x_m)+d(x_m,b)<\frac{d(a,b)}{2}+\frac{d(a,b)}{2}=d(a,b)$, giving the desired contradiction.

To prove Part 4, suppose that $(x_n)_{n\geq p}$ converges, say $x_n\to a$ in X. Choose $m\in\mathbb{Z}_{\geq p}$ such that when $n\geq m$ we have $d(x_n,a)<1$. Then for all $n\in\mathbb{Z}_{\geq p}$ we have $d(x_n,a)\leq r$ where $r=\max\big\{d(x_p,a),d(x_{p+1},a),\cdots,d(x_{m-1},a),1\big\}$ so that $x_n\in B(a,r{+}1)$.

To prove Part 5, note that since $d(x_n,a)\geq 0$ we have $d(x_n,a)=\big|d(x_n,a)-0\big|$ and so

$$\begin{aligned}\lim_{n\to\infty} x_n=a\text{ in }X &\iff \forall\epsilon>0\ \exists m\in\mathbb{Z}_{\geq p}\ \forall n\in\mathbb{Z}_{\geq p}\ d(x_n,a)<\epsilon\\ &\iff \forall\epsilon>0\ \exists m\in\mathbb{Z}_{\geq p}\ \forall n\in\mathbb{Z}_{\geq p}\ \big|d(x_n,a)-0\big|<\epsilon\\ &\iff \lim_{n\to\infty} d(x_n,a)=0\text{ in }\mathbb{R}.\end{aligned}$$

5.3 Note: Because of Part 2 of the above theorem, the initial index p of a sequence $(x_n)_{n\geq p}$ does not effect whether or not the sequence converges and it does not effect the limit. For this reason, we often omit the initial index p from our notation and write (x_n) for the sequence $(x_n)_{n\geq p}$. Also, we often choose a specific value of p, usually $p=1$, in the statements or the proofs of various theorems with the understanding that any other initial value would work just as well.

5.4 Theorem: *(Components of Sequences in $\mathbb{R}^m$) Let $(x_n)_{n\geq 1}$ be a sequence in $\mathbb{R}^m$, say $x_n = \big(x_{n,1}, x_{n,2}, \cdots, x_{n,m}\big) \in \mathbb{R}^m$, and let $a = (a_1, a_2, \cdots, a_m) \in \mathbb{R}^m$. Then using any of the metrics d_1, d_2 or d_∞ in $\mathbb{R}^m$, we have $\lim_{n\to\infty} x_n = a$ in $\mathbb{R}^m$ if and only if $\lim_{n\to\infty} x_{n,k} = a_k$ in $\mathbb{R}$ for all indices k with $1 \leq k \leq m$.*

Proof: Let $p = 1$, 2 or ∞. Suppose that $\lim_{n\to\infty} x_n = a$ in $\mathbb{R}^m$. Let $1 \leq k \leq m$ and let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ such that when $n \geq m$ we have $d_p(x_n, a) < \epsilon$, that is $\|x_n - p\|_p < \epsilon$. Then when $n \geq m$ we have

$$|x_{n,k} - a_k| = \big(|x_{n,k} - a_k|^p\big)^{1/p} \leq \Big(\sum_{j=1}^{m} |x_{n,j} - a_j|^p\Big)^{1/p} = \|x_n - a\|_p < \epsilon\,,$$

and so $x_{n,k} \to a_k$ in $\mathbb{R}$, as required.

Suppose, conversely, that for all k with $1 \leq k \leq m$ we have $\lim_{n\to\infty} x_{n,k} = a_k$ in $\mathbb{R}$. Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ such that for all $n \geq m$ we have $|x_{n,k} - a_k| < \frac{\epsilon}{m}$ for $1 \leq k \leq m$. Then, letting e_k denote the k^{th} standard basis vector in $\mathbb{R}^m$, when $n \geq m$ we have

$$\begin{aligned}\|x_n - a\|_p &= \Big\|\sum_{k=1}^{m} (x_{n,k} - a_k)\, e_k\Big\|_p \leq \sum_{k=1}^{m} \big\|(x_{n,k} - a_k)e_k\big\|_p \\ &= \sum_{k=1}^{m} \big|x_{n,k} - a_k\big|\, \|e_k\|_p = \sum_{k=1}^{m} \big|x_{n,k} - a_k\big| < \sum_{k=1}^{m} \tfrac{\epsilon}{m} = \epsilon\end{aligned}$$

so that $x_n \to a$ in $\mathbb{R}^m$, as required.

5.5 Note: When $(x_n)_{n\geq 1}$ is a sequence in $\mathbb{R}^\infty$, ℓ_1, ℓ_2 or ℓ_∞, each term x_n is itself a sequence (so that (x_n) is a sequence of sequences) and we can write $x_n = (x_{n,k})_{k\geq 1}$. We have sequences $x_1 = (x_{1,1}, x_{1,2}, \cdots)$, $x_2 = (x_{2,1}, x_{2,2}, x_{2,3}, \cdots)$, and $x_3 = (x_{3,1}, x_{3,2}, \cdots)$ and so on. This is not the same thing as a subsequence (x_n), which is a single sequence $(x_{n_k})_{k\geq 1} = (x_{n_1}, x_{n_2}, x_{n_3}, \cdots)$.

5.6 Theorem: $\big($*Components of Sequences in ℓ_p*$\big)$. *Let $p = 1$, 2 or ∞. Let $(x_n)_{n\geq 1}$ be a sequence in ℓ_p, say $x_n = (x_{n,k})_{k\geq 1} \in \ell_p$, and let $a = (a_k)_{k\geq 1} \in \ell_p$. If $\lim_{n\to\infty} x_n = a$ in ℓ_p then $\lim_{n\to\infty} x_{n,k} = a_k$ in $\mathbb{R}$ for all $k \in \mathbb{Z}^+$.*

Proof: The proof is the same as the first half of the proof of Theorem 5.4. Suppose that $\lim_{n\to\infty} x_n = a$ in ℓ_p. Let $k \in \mathbb{Z}^+$ and let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ such that when $n \geq m$ we have $\|x_n - a\|_p < \epsilon$. Then when $n \geq m$ we have

$$|x_{n,k} - a_k| = \big(|x_{n,k} - a_k|^p\big)^{1/p} \leq \Big(\sum_{j=1}^{\infty} |x_{n,j} - a_j|^p\Big)^{1/p} = \|x_n - a\|_p < \epsilon$$

and so $x_{n,k} \to a_k$ in $\mathbb{R}$, as required.

5.7 Note: Unlike the case in $\mathbb{R}^m$, in the infinite-dimensional spaces ℓ_p, when $x_{n,k} \to a_k$ in $\mathbb{R}$ for all indices k, it does not necessarily follow that $x_n \to a$ in ℓ_p. For example, you can verify, as an exercise, that when $x_n = e_n$ (the n^{th} standard basis vector in $\mathbb{R}^\infty$), we have $\lim_{n\to\infty} x_{n,k} = 0$ in $\mathbb{R}$ for all $k \in \mathbb{Z}^+$, but $\lim_{n\to\infty} x_n \neq 0$ in ℓ_p.

5.8 Exercise: For each $n \in \mathbb{Z}^+$, let $x_n \in \mathbb{R}^\infty$ be the sequence given by $x_n = \frac{1}{n}\sum_{k=1}^{n} e_k$, that is by $x_n = (x_{n,k})_{k\geq 1} = \big(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}, 0, 0, 0, \cdots\big)$ with n non-zero terms. Show that (x_n) converges in $(\mathbb{R}^\infty, d_2)$ but diverges in $(\mathbb{R}^\infty, d_1)$.

5.9 Remark: In the definition of the limit of a sequence in a metric space X (Definition 5.1), we can replace the strict inequality $d(x_n, a) < \epsilon$ by the inequality $d(x_n, a) \le \epsilon$ without changing the meaning. In other words, for a sequence $(x_n)_{n \ge p}$ in X and an element $a \in X$ we have

$$\lim_{n \to \infty} x_n = a \text{ in } X \iff \forall \epsilon > 0 \ \exists m \in \mathbb{Z}_{\ge p} \forall n \in \mathbb{Z}_{\ge p} \big(n \ge m \implies d(x_n, a) \le \epsilon \big).$$

Here is a proof. Suppose that $\lim\limits_{n \to \infty} x_n = a$ in X. Let $\epsilon > 0$. Choose $m \in \mathbb{Z}_{\ge p}$ so that when $n \ge m$ we have $d(x_n, a) < \epsilon$. Then when $n \ge m$ we also have $d(x_n, a) \le \epsilon$. Suppose, conversely, that for every $\epsilon > 0$ there exists $m \in \mathbb{Z}_{\ge p}$ such that whenever $n \ge m$ we have $d(x_n, a) \le \epsilon$. Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ such that when $n \ge m$ we have $d(x_n, a) \le \frac{\epsilon}{2}$. Then when $n \ge m$ we have $d(x_n, a) \le \frac{\epsilon}{2} < \epsilon$.

5.10 Remark: Note that for $h \in \mathcal{B}[a, b]$ and $r > 0$, we have

$$\|h\|_\infty \le r \iff \sup \big\{ |h(x)| \,\big|\, a \le x \le b \big\} \le r \iff |h(x)| \le r \text{ for all } x \in [a, b].$$

We also remark that we would not have equivalence if we replaced $\le r$ by $< r$, as we only have a one way implication: if $|h(x)| < r$ for all $x \in [a, b]$ then $\sup \big\{ |h(x)| \,\big|\, a \le x \le b \big\} \le r$.

5.11 Theorem: *(Limits in $\mathcal{B}[a, b]$ and Uniform Convergence) Let $(f_n)_{n \ge 1}$ be a sequence in $\mathcal{B}[a, b]$, and let $g \in \mathcal{B}[a, b]$. Then $f_n \to g$ in $\big(\mathcal{B}[a, b], d_\infty\big)$ if and only if $f_n \to g$ uniformly on $[a, b]$.*

Proof: This follows immediately from the above two remarks. Indeed we have

$$\begin{aligned} f_n \to g \text{ in } \mathcal{B}[a, b] &\iff \forall \epsilon > 0 \ \exists m \in \mathbb{Z}^+ \ \forall n \in \mathbb{Z}^+ \big(n \ge m \implies \|f_n - g\|_\infty \le \epsilon \big) \\ &\iff \forall \epsilon > 0 \ \exists m \in \mathbb{Z}^+ \ \forall n \in \mathbb{Z}^+ \big(n \ge m \implies \big|f_n(x) - g(x)\big| \le \epsilon \text{ for all } x \in [a, b] \big) \\ &\iff f_n \to g \text{ uniformly on } [a, b]. \end{aligned}$$

5.12 Remark: In Chapter 3, we discussed pointwise convergence and uniform convergence of sequences of functions. In this Chapter, we are discussing convergence in a metric space. For a metric space X whose elements are functions, such as $\mathcal{B}[a, b]$ or $\mathcal{C}[a, b]$, a sequence in X is a sequence of functions, so we now have several different notions of convergence for sequences of functions. The above theorem shows that convergence in the metric space $\mathcal{B}[a, b]$ $\big($hence also in $\mathcal{C}[a, b]\big)$ using the supremum metric d_∞, is the same thing as uniform convergence. One might ask whether convergence in $\mathcal{C}[a, b]$ using the metrics d_1 or d_2 implies, or is implied by, pointwise convergence. The answer is negative, as the following exercises illustrate.

5.13 Exercise: Define $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = 1 - nx$ for $0 \le x \le \frac{1}{n}$ and $f_n(x) = 0$ for $\frac{1}{n} \le x \le 1$. Show that $f_n \to 0$ in $\mathcal{C}[0, 1]$ using either of the metrics d_1 or d_2, but $f_n \not\to 0$ pointwise on $[0, 1]$.

5.14 Exercise: Define $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = n^2 x - n^3 x^2$ for $0 \le x \le \frac{1}{n}$ and $f_n(x) = 0$ for $\frac{1}{n} \le x \le 1$. Show that $f_n \to 0$ pointwise on $[0, 1]$ but $f_n \not\to 0$ in $\mathcal{C}[0, 1]$ using either of the metrics d_1 or d_2.

5.15 Exercise: Define $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = \sqrt{n}\, x^n$. Show that $(f_n)_{n \ge 1}$ converges in $\big(\mathcal{C}[0, 1], d_1\big)$ but diverges in $\big(\mathcal{C}[0, 1], d_2\big)$.

Limits and Closed Sets

5.16 Theorem: *(The Sequential Characterization of Limit Points and Closed Sets) Let X be a metric space, let $a \in X$, and let $A \subseteq X$.*

(1) $a \in A'$ if and only if there exists a sequence (x_n) in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$ in X.

(2) $a \in \overline{A}$ if and only if there exists a sequence (x_n) in A with $\lim\limits_{n\to\infty} x_n = a$ in X.

(3) A is closed in X if and only if for every sequence (x_n) in A which converges in X, we have $\lim\limits_{n\to\infty} x_n \in A$.

Proof: We prove Parts 1 and 3 and leave the proof of Part 2 as an exercise. Suppose that $a \in A'$ (which means that for every $r > 0$ we have $B^*(a,r) \cap A \neq \emptyset$). For each $n \in \mathbb{Z}^+$, choose $x_n \in B^*\left(a, \frac{1}{n}\right) \cap A$, that is choose $x_n \in A \setminus \{a\}$ with $d(x_n, a) < \frac{1}{n}$. Then $(x_n)_{n\geq 1}$ is a sequence in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$.

Suppose, conversely, that $(x_n)_{n\geq 1}$ is a sequence in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$. Let $r > 0$. Choose $m \in \mathbb{Z}^+$ such that $d(x_n, a) < r$ for all $n \geq m$. Since $x_m \in A \setminus \{a\}$ with $d(x_m, a) < r$, we have $x_m \in B^*(a,r) \cap A$ and so $B^*(a,r) \cap A \neq \emptyset$. This proves Part 1.

To prove Part 3, suppose that A is closed in X. Let $(x_n)_{n\geq 1}$ be a sequence in A which converges in X, and let $a = \lim\limits_{n\to\infty} x_n \in X$. Suppose, for a contradiction, that $a \notin A$. Since $a \notin A$ we have $A = A \setminus \{a\}$ so in fact (x_n) is a sequence in $A \setminus \{a\}$. Since (x_n) is a sequence in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$, it follows from Part 1 that $a \in A'$. Since A is closed we have $A' \subseteq A$ and so $a \in A$ giving the desired contradiction.

Suppose, conversely, that for every sequence in A which converges in X, the limit of the sequence lies in A. Let $a \in A'$. By Part 1, we can choose a sequence (x_n) in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$. Then (x_n) is a sequence in A which converges in X, so its limit lies in A, that is $a \in A$. Since $a \in A'$ was arbitrary, this shows that $A' \subseteq A$, and so A is closed. This proves Part 3.

5.17 Example: Let U be a normed linear space, let $a \in U$ and let $r > 0$. Show that $\overline{B(a,r)} = \overline{B}(a,r)$ (so the closed ball is equal to the closure of the open ball).

Solution: We saw, in Example 4.32, that $\overline{B}(a,r)$ is closed. Since $\overline{B}(a,r)$ is closed and $B(a,r) \subseteq \overline{B}(a,r)$, it follows that $\overline{B(a,r)} \subseteq \overline{B}(a,r)$. Let $b \in \overline{B}(a,r)$, that is let $b \in U$ with $\|b - a\| \leq r$. If $\|b - a\| < r$ then we have $b \in B(a,r) \subseteq \overline{B(a,r)}$. Suppose that $\|b - a\| = r$. For $n \in \mathbb{Z}^+$, let $x_n = a + \left(1 - \frac{1}{n}\right)(b - a) \in U$. Note that

$$\|x_n - a\| = \left\|\left(1 - \tfrac{1}{n}\right)(b - a)\right\| = \left(1 - \tfrac{1}{n}\right)\|b - a\| = \left(1 - \tfrac{1}{n}\right)r < r$$

so that $x_n \in B(a,r)$. Note that

$$\|x_n - b\| = \left\|\tfrac{1}{n}(a - b)\right\| = \tfrac{1}{n}\|a - b\| = \tfrac{r}{n} \to 0 \text{ in } \mathbb{R}$$

so that we have $x_n \to b$ in U (by Part 6 of Theorem 5.2). Since (x_n) is a sequence in $B(a,r)$ with $x_n \to b$ in U, it follows that $b \in \overline{B(a,r)}$ by Part 2 of the above theorem.

5.18 Example: In the previous example, it might have seemed intuitively obvious that $\overline{B(a,r)} = \overline{B}(a,r)$, but in fact this is not true in all metric spaces. For example in $\mathbb{Z}$ (using the same standard metric used in $\mathbb{R}$) we have $B(0,1) = \{0\}$ and $\overline{B(0,1)} = \{0\}$, but $\overline{B}(0,1) = \{-1, 0, 1\}$.

5.19 Exercise: Recall that $\mathbb{R}^\infty \subseteq \ell_1 \subseteq \ell_2 \subseteq \ell_\infty$. Determine whether $\mathbb{R}^\infty$ is closed in ℓ_1 using the metric d_1 (this problem is solved in Example 5.20, below). Determine which of the spaces $\mathbb{R}^\infty$ and ℓ_1 is closed in ℓ_2 using the metric d_2. Determine which of the spaces $\mathbb{R}^\infty$, ℓ_1 and ℓ_2 is closed in ℓ_∞ using the metric d_∞.

5.20 Example: Show that $\mathbb{R}^\infty$ is dense in the metric space (ℓ_1, d_1).

Solution: Since the closure of $\mathbb{R}^\infty$ in ℓ_1 is contained in ℓ_1 (by the definition of closure), it suffices to show that $\ell_1 \subseteq \overline{\mathbb{R}^\infty}$. Let $a = (a_n)_{n\geq 1} \in \ell_1$, so we have $\sum\limits_{n=1}^{\infty} |a_n| < \infty$. For each $n \in \mathbb{Z}^+$ let $x_n = (x_{n,k})_{k\geq 1}$ be the sequence given by $x_n = \sum\limits_{k=1}^{n} a_k e_k$ (where e_k is the k^{th} standard basis vector in $\mathbb{R}^\infty$), that is

$$(x_{n,k})_{k\geq 1} = (x_{n,1}, x_{n,2}, x_{n,3}, \cdots) = (a_1, a_2, \cdots, a_n, 0, 0, 0, \cdots).$$

Then each $x_n \in \mathbb{R}^\infty$ and, in the metric space ℓ_1, we have $x_n \to a$ because given $\epsilon > 0$ we can choose an index m so that $\sum\limits_{k>m} |a_k| < \epsilon$ and then for all $n \geq m$ we have

$$\|x_n - a\|_1 = \sum_{k=1}^{\infty} |x_{n,k} - a_k| = \sum_{k>n} |a_k| \leq \sum_{k>m} |a_k| < \epsilon.$$

It follows, from Part 2 of Theorem 5.16, that $a \in \overline{\mathbb{R}^\infty}$, as required.

5.21 Exercise: Let

$$\begin{aligned}
\mathcal{R}[a,b] &= \big\{ f \in \mathcal{B}[a,b] \,\big|\, f \text{ is Riemann integrable} \big\}, \\
\mathcal{P}[a,b] &= \big\{ f \in \mathcal{B}[a,b] \,\big|\, f \text{ is a polynomial} \big\}, \\
\mathcal{C}^1[a,b] &= \big\{ f \in \mathcal{B}[a,b] \,\big|\, f \text{ is continuously differentiable} \big\}.
\end{aligned}$$

Note that

$$\mathcal{P}[a,b] \subseteq \mathcal{C}^1[a,b] \subseteq \mathcal{C}[a,b] \subseteq \mathcal{R}[a,b] \subseteq \mathcal{B}[a,b].$$

Determine which of the above spaces are closed in the metric space $\mathcal{B}[a,b]$, using the supremum metric d_∞ (we deal with the space $\mathcal{C}[a,b]$ in the following example).

5.22 Example: Show that $\mathcal{C}[a,b]$ is closed in the metric space $\big(\mathcal{B}[a,b], d_\infty\big)$.

Solution: Let (f_n) be a sequence in $\mathcal{C}[a,b]$ which converges in $\mathcal{B}[a,b]$ (using the metric d_∞). Let $g = \lim\limits_{n\to\infty} f_n$ in $\mathcal{B}[a,b]$. By Theorem 5.11, we know that $f_n \to g$ uniformly on $[a,b]$. Since each function f_n is continuous, it follows from Theorem 3.9 (Uniform Convergence and Continuity) that g is continuous, that is $g \in \mathcal{C}[a,b]$. By the Sequential Characterization of Closed Sets (Part 3 of Theorem 5.16), it follows that $\mathcal{C}[a,b]$ is closed in $\mathcal{B}[a,b]$.

5.23 Definition: Let (X, d_X) and (Y, d_Y) be metric spaces. Let $A \subseteq X$, let $f : A \to Y$, let $a \in A'$, and let $b \in Y$. We say that the **limit** of $f(x)$ as x tends to a is equal to b, and we write $\lim\limits_{x \to a} f(x) = b$, when

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \in A \ \big(0 < d_X(x, a) < \delta \Longrightarrow d_Y\big(f(x), b\big) < \epsilon\big).$$

5.24 Theorem: *(The Sequential Characterization of Limits) Let X and Y be metric spaces, let $A \subseteq X$, let $f : A \to Y$, let $a \in A' \subseteq X$, and let $b \in Y$. Then $\lim\limits_{x \to a} f(x) = b$ if and only if for every sequence (x_n) in $A \setminus \{a\}$ with $x_n \to a$ we have $\lim\limits_{n \to \infty} f(x_n) = b$.*

Proof: Suppose that $\lim\limits_{x \to A} f(x) = b$. Let (x_n) be a sequence in $A \setminus \{a\}$ with $x_n \to a$. Let $\epsilon > 0$. Since $\lim\limits_{x \to a} f(x) = b$ we can choose $\delta > 0$ such that $0 < d(x, a) < \delta \Longrightarrow d\big(f(x), b\big) < \epsilon$. Since $x_n \to a$ we can choose $m \in \mathbb{Z}^+$ such that $n \geq m \Longrightarrow d(x_n, a) < \delta$. For $n \geq m$ we have $d(x_n, a) < \delta$ and we have $x_n \neq a$ $\big($since (x_n) is a sequence in $A \setminus \{a\}\big)$, so that $0 < d(x_n, a) < \delta$, and hence $d\big(f(x_n), b\big) < \epsilon$. Thus $\lim\limits_{n \to \infty} f(x_n) = b$, as required.

Suppose, conversely, that $\lim\limits_{x \to a} f(x) \neq b$. Choose $\epsilon > 0$ such that for every $\delta > 0$ there exists $x \in A$ such that $0 < d(x, a) < \delta$ and $d\big(f(x), b\big) \geq \epsilon$. For each $n \in \mathbb{Z}^+$, choose $x_n \in A$ such that $0 < d(x_n, a) < \frac{1}{n}$ and $d\big(f(x_n), b\big) \geq \epsilon$. For each n, since $0 < d(x_n, a)$ we have $x_n \neq a$ so the sequence (x_n) lies in $A \setminus \{a\}$. Since $d(x_n, a) < \frac{1}{n}$ for all $n \in \mathbb{Z}^+$, it follows that $x_n \to a$. Since $d\big(f(x_n), b\big) \geq \epsilon$ for all $n \in \mathbb{Z}^+$, it follows that $\lim\limits_{n \to \infty} f(x) \neq b$. Thus we have found a sequence (x_n) in $A \setminus \{a\}$ with $x_n \to a$ such that $\lim\limits_{n \to \infty} f(x_n) \neq b$.

5.25 Definition: Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$. For $a \in X$, we say that f is **continuous** at a when for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$, if $d_X(x, a) < \delta$ then $d_Y\big(f(x), f(a)\big) < \epsilon$. We say that f is **continuous** (on X) when f is continuous at every point $a \in X$. We say that f is **uniformly continuous** (on X) when for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$, if $d_X(x, y) < \delta$ then $d_Y\big(f(x), f(y)\big) < \epsilon$. We say that f is **Lipschitz continuous** (on X) when there is a constant $\ell \geq 0$, called a **Lipschitz constant** for f, such that for all $x, y \in X$ we. have $d\big(f(x), f(y)\big) \leq \ell \cdot d(x, y)$. Note that if f is Lipschitz continuous then f is also uniformly continuous (indeed we can take $\delta = \frac{\epsilon}{\ell}$ in the definition of uniform continuity). A bijective map $f : X \to Y$ such that both f and f^{-1} are continuous is called a **homeomorphism**.

5.26 Note: Let X and Y be metric spaces and let $a \in X$. If a is a limit point of X then f is continuous at a if and only if $\lim\limits_{x \to a} f(x) = f(a)$. If a is an isolated point of X then f is necessarily continuous at a, vacuously.

5.27 Theorem: *(The Sequential Characterization of Continuity) Let X and Y be metric spaces using metrics d_X and d_Y, let $f : X \to Y$, and let $a \in X$. Then f is continuous at a if and only if for every sequence (x_n) in X with $x_n \to a$ we have $\lim\limits_{n \to \infty} f(x_n) = f(a)$.*

Proof: The proof is left as an exercise.

5.28 Exercise: Let X, Y and Z be metric spaces, let $f : X \to Y$, let $g : Y \to Z$. Show that if f is continuous at the point $a \in X$ and g is continuous at the point $f(a) \in Y$ then the composite function $g \circ f$ is continuous at a.

5.29 Theorem: *(The Topological Characterization of Continuity) Let X and Y be metric spaces and let $f : X \to Y$. Then*

(1) f is continuous (on X) if and only if $f^{-1}(V)$ is open in X for every open set V in Y,
(2) f is continuous (on X) if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.

Proof: To prove Part 1, suppose that f is continuous in X. Let V be open in Y. Let $a \in f^{-1}(V)$ and let $f(a) \in V$. Since V is open, we can choose $\epsilon > 0$ such that $B\big(f(a), \epsilon\big) \subseteq V$. Since f is continuous at a we can choose $\delta > 0$ such that for all $x \in X$ with $d(x, a) < \delta$ we have $d\big(f(x), f(a)\big) < \epsilon$. Then we have $f\big(B(a, \delta)\big) \subseteq B\big(f(a), \epsilon\big) \subseteq V$ and so $B(a, \delta) \subseteq f^{-1}(V)$. Thus $f^{-1}(V)$ is open in X, as required.

Suppose, conversely, that $f^{-1}(V)$ is open in X for every open set V in Y. Let $a \in X$ and let $\epsilon > 0$. Taking $V = B\big(f(a), \epsilon\big)$, which is open in Y, we see that $f^{-1}\big(B\big(f(a), \epsilon\big)\big)$ is open in X. Since $a \in f^{-1}\big(B(f(a), \epsilon\big)$ and $f^{-1}\big(B\big(f(a), \epsilon\big)\big)$ is open in X, we can choose $\delta > 0$ such that $B(a, \delta) \subseteq f^{-1}\big(B\big(f(a), \epsilon\big)\big)$. Then we have $f\big(B(a, \delta)\big) \subseteq B\big(f(a), \epsilon\big)$ or, in other words, for all $x \in X$, if $d(x, a) < \delta$ then $d\big(f(x), f(a)\big) < \epsilon$. Thus f is continuous at a hence, since a was arbitrary, f is continuous on X.

This completes the proof of Part 1, and Part 2 follows by taking complements since for every set $B \subseteq Y$ we have $\big(f^{-1}(B)\big)^c = f^{-1}(B^c)$. Indeed for all $x \in A$ we have

$$x \in \big(f^{-1}(B)\big)^c \iff x \notin f^{-1}(B) \iff f(x) \notin B \iff f(x) \in B^c \iff x \in f^{-1}(B^c).$$

5.30 Definition: Let X and Y be topological spaces and let $f : X \to Y$. We say that f is **continuous** (on X) when $f^{-1}(V)$ is open in X for every open set V in Y. A bijective map $f : X \to Y$ such that both f and f^{-1} are continuous is called a **homeomorphism**.

5.31 Theorem: *(Composition of Continuous Functions) Let X, Y and Z be metric spaces (or topological spaces), let $f : X \to Y$, and let $g : Y \to Z$. If f and g are continuous then the composite function $g \circ f : X \to Z$ is continuous.*

Proof: Let $h = g \circ f : X \to Z$. If $W \subseteq Z$ is open in Z, then $g^{-1}(W)$ is open in Y (since g is continuous), hence $h^{-1}(W) = f^{-1}\big(g^{-1}(W)\big)$ is open in X (since f is continuous). Thus h is continuous, by Theorem 5.29 (or by Definition 5.30)

5.32 Example: Let $A = \big\{(x, y) \in \mathbb{R}^2 \,\big|\, y < x^2\big\}$. Show that A is open in $\mathbb{R}^2$.

Solution: We remark that it is surprisingly difficult to show that A is open directly from the definition of an open set (as mentioned in Remark 4.34). But we can make use of the Topological Characterization of Continuity to give a quick proof. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = y - x^2$. Note that f is continuous (polynomial functions, and indeed all elementary functions, are continuous) and we have $A = \big\{(x, y) \,\big|\, f(x, y) < 0\big\} = f^{-1}(B)$ where B is the open interval $(-\infty, 0)$. Since B is open in $\mathbb{R}$ and f is continuous, it follows that $A = f^{-1}(B)$ is open in $\mathbb{R}^2$.

5.33 Example: Recall from Example 4.41 that every set $U \subseteq \mathcal{C}[a, b]$ which is open using the metric d_1 is also open using the metric d_∞, but not vice versa. It follows (from Theorem 5.29) that the identity map $I : \mathcal{C} \to \mathcal{C}[a, b]$ given by $I(f) = f$ is continuous as a map from the metric space $\big(\mathcal{C}[a, b], d_\infty\big)$ to the metric space $\big(\mathcal{C}[a, b], d_1\big)$, but not vice versa.

Continuity of Linear Maps

5.34 Note: If U and V are inner product spaces and $L : U \to V$ is an inner product space isomorphism, then L and its inverse preserve distance so they are both continuous (we can take $\delta = \epsilon$ in the definition of continuity), hence L is a homeomorphism.

If U and V are finite-dimensional inner product spaces with say $\dim U = n$ and $\dim V = m$, and if $\phi : \mathbb{R}^n \to U$ and $\psi : \mathbb{R}^m \to V$ are inner product space isomorphisms (obtained by choosing orthonormal bases for U and V) then a map $F : U \to V$ is continuous if and only if the composite map $\psi^{-1} F \phi : \mathbb{R}^n \to \mathbb{R}^m$ is continuous. In particular, if F is linear then F is continuous (since $\psi^{-1} F \phi : \mathbb{R}^n \to \mathbb{R}^m$ is linear, hence continuous).

We shall see below (in Corollary 5.39) that the same is true for finite dimensional normed linear spaces: every linear map between finite dimensional normed linear spaces is continuous. But this is not always true (see Example 5.33) for infinite dimensional spaces.

5.35 Theorem: *Let U and V be normed linear spaces and let $F : U \to V$ be a linear map. Then the following are equivalent:*

(1) F is Lipschitz continuous on U,
(2) F is continuous at some point $a \in U$,
(3) F is continuous at 0, and
(4) $F\big(\overline{B}(0,1)\big)$ is bounded.

In this case, if $m \geq 0$ with $F\big(\overline{B}(0,1)\big) \subseteq B(0,m)$ then m is a Lipschitz constant for F.

Proof: It is clear that if F is Lipschitz continuous on U then F is continuous at some point $a \in U$ (indeed F is continuous at every point $a \in U$). Let us show that if F is continuous at some point $a \in U$ then F is continuous at 0. Suppose that F is continuous at $a \in U$. Let $\epsilon > 0$. Since F is continuous at $a \in U$, we can choose $\delta_1 > 0$ such that for all $u \in U$ we have $\|u - a\| \leq \delta_1 \Longrightarrow \big\|F(u) - F(a)\big\| \leq 1$. Choose $\delta = \delta_1 \epsilon$. Let $x \in U$ with $\|x - 0\| < \delta$. If $x = 0$ then $\big\|F(x) - F(0)\big\| = \|0\| = 0$. Suppose that $x \neq 0$. Then for $u = a + \frac{\delta_1 x}{\|x\|}$ we have $\|u - a\| = \big\|\frac{\delta_1 x}{\|x\|}\big\| = \delta_1$ and so $\big\|F(u-a)\big\| = \big\|F(u) - F(a)\big\| \leq 1$, that is $\big\|F\big(\frac{\delta_1 x}{\|x\|}\big)\big\| \leq 1$ hence, by the linearity of F and the scaling property of the norm, we have

$$\big\|F(x) - F(0)\big\| = \big\|F(x)\big\| = \tfrac{\|x\|}{\delta_1}\big\|F\big(\tfrac{\delta_1 x}{\|x\|}\big)\big\| \leq \tfrac{\|x\|}{\delta_1} < \tfrac{\delta_1 \epsilon}{\delta_1} = \epsilon.$$

Thus F is continuous at 0, as required

Next we show that if F is continuous at 0 then $F\big(\overline{B}(0,1)\big)$ is bounded. Suppose that F is continuous at 0. Choose $\delta > 0$ so that for all $u \in U$ we have $\|u\| \leq \delta \Longrightarrow \|F(u)\| \leq 1$. Let $m = \frac{1}{\delta}$. For $x \in U$, when $x = 0$ we have $\|F(x)\| = 0 \leq m$ and when $0 < \|x\| \leq 1$ we have

$$\|F(x)\| = \Big\|\tfrac{\|x\|}{\delta}\, F\big(\tfrac{\delta x}{\|x\|}\big)\Big\| = \tfrac{\|x\|}{\delta}\Big\|F\big(\tfrac{\delta x}{\|x\|}\big)\Big\| \leq \tfrac{\|x\|}{\delta} = m\|x\| \leq m.$$

Thus $F\big(\overline{B}(0,1)\big)$ is bounded, as required.

Finally we show that if $F\big(\overline{B}(0,1)\big)$ is bounded then F is Lipschitz continuous. Suppose that $F\big(\overline{B}(0,1)\big)$ is bounded. Choose $m > 0$ so that $\|F(u)\| \leq m$ for all $u \in U$ with $\|u\| \leq 1$. Let $x, y \in U$. If $x = y$ then $\big\|F(x) - F(y)\big\| = 0$. Suppose that $x \neq y$. Then we have $\big\|\frac{x-y}{\|x-y\|}\big\| = 1$ so that $\big\|F\big(\frac{x-y}{\|x-y\|}\big)\big\| \leq m$ and so

$$\big\|F(x) - F(y)\big\| = \big\|F(x-y)\big\| = \|x-y\|\big\|F\big(\tfrac{x-y}{\|x-y\|}\big)\big\| \leq m\|x-y\|.$$

Thus F is Lipschitz continuous with Lipschitz constant m, as required.

5.36 Example: Define $L : \big(\mathcal{C}[a,b], d_\infty\big) \to \big(\mathcal{C}[a,b], d_\infty\big)$ by $L(f) = \displaystyle\int_a^x f(t)\,dt$. Show that L is Lipschitz continuous.

Solution: Let $f \in \mathcal{C}[a,b]$ with $\|f\|_\infty \le 1$, that is with $\max\limits_{a\le x\le b} |f(x)| \le 1$. Then

$$\big\|F(f)\big\|_\infty = \max_{a\le x\le b}\left|\int_a^x f(t)\,dt\right| \le \max_{a\le x\le b}\int_a^x 1\,dt = \max_{a\le x\le b}|x-a| = |b-a|.$$

Thus $F\big(\,\overline{B}(0,1)\big)$ is bounded and so F is uniformly continuous.

5.37 Example: Define $D : \big(\mathcal{C}^1[0,1], d_\infty\big) \to \mathcal{C}[0,1], d_\infty\big)$ by $D(f) = f'$. Show that D is not continuous.

Solution: For $n \in \mathbb{Z}^+$, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$. Then $f_n \in \mathcal{C}^1[a,b]$, and $\|f_n\|_\infty = \max\limits_{0\le x\le 1}|x^n| = 1$ so that $f_n \in B(0,1)$, and $\big\|D(f_n)\big\|_\infty = \max\limits_{0\le x\le 1}\big|n\,x^{n-1}\big| = n$. Thus $D\big(\,\overline{B}(0,1)\big)$ is not bounded, so D is not continuous $\big($at any point $g \in \mathcal{C}[0,1]\big)$.

5.38 Theorem: *Let U be an n-dimensional normed linear space over $\mathbb{R}$. Let $\{u_1,\cdots,u_n\}$ be any basis for U and let $\phi : \mathbb{R}^n \to U$ be the associated vector space isomorphism given by $\phi(t) = \sum\limits_{k=1}^n t_k u_k$. Then both ϕ and ϕ^{-1} are Lipschitz continuous.*

Proof: Let $M = \Big(\sum\limits_{k=1}^n \|u_k\|^2\Big)^{1/2}$. For $t \in \mathbb{R}^n$ we have

$$\begin{aligned}\big\|\phi(t)\big\| &= \Big\|\sum_{k=1}^n t_k u_k\Big\| \le \sum_{k=1}^n |t_k|\,\|u_k\| \quad \text{, by the Triangle Inequality,}\\ &\le \Big(\sum_{k=1}^n {t_k}^2\Big)^{1/2}\Big(\sum_{k=1}^n \|u_k\|^2\Big)^{1/2} \quad \text{, by the Cauchy-Schwarz Inequality,}\\ &= M\|t\|.\end{aligned}$$

For all $s,t \in \mathbb{R}^n$, $\big\|\phi(s)-\phi(t)\big\| = \big\|\phi(s-t)\big\| \le M\,\|s-t\|$, so ϕ is Lipschitz continuous.

Note that the map $N : U \to \mathbb{R}$ given by $N(x) = \|x\|$ is (uniformly) continuous, indeed we can take $\delta = \epsilon$ in the definition of continuity. Since ϕ and N are both continuous, so is the composite $G = N \circ \phi : \mathbb{R}^n \to \mathbb{R}$, which given by $G(t) = \big\|\phi(t)\big\|$. By the Extreme Value Theorem, the map G attains its minimum value on the unit sphere $\big\{t \in \mathbb{R}^n \big| \|t\| = 1\big\}$, which is compact. Let $m = \min\limits_{\|t\|=1} G(t) = \min\limits_{\|t\|=1}\big\|\phi(t)\big\|$. Note that $m > 0$ because when $t \ne 0$ we have $\phi(t) \ne 0$ (since ϕ is a bijective linear map) and hence $\|\phi(t)\| \ne 0$. For $t \in \mathbb{R}^n$, if $\|t\| > 1$ then we have $\big\|\frac{t}{\|t\|}\big\| = 1$ so, by the choice of m,

$$\big\|\phi(t)\big\| = \|t\|\,\Big\|\phi\big(\tfrac{t}{\|t\|}\big)\Big\| \ge \|t\|\cdot m > m.$$

It follows that for all $t \in \mathbb{R}^n$, if $\big\|\phi(t)\big\| \le m$ then $\|t\| \le 1$. Since ϕ is bijective, it follows that for $x \in U$, if $\|x\| \le m$ then $\big\|\phi^{-1}(x)\big\| \le 1$. Thus for all $x \in U$, if $x = 0$ then $\|\phi^{-1}(x)\| = 0 = \frac{\|x\|}{m}$ and if $x \ne 0$ then since $\big\|\frac{mx}{\|x\|}\big\| = m$ we have

$$\big\|\phi^{-1}(x)\big\| = \tfrac{\|x\|}{m}\,\big\|\phi^{-1}\big(\tfrac{mx}{\|x\|}\big)\big\| \le \tfrac{\|x\|}{m}.$$

For all $x,y \in U$, we have $\big\|\phi^{-1}(x)-\phi^{-1}(y)\big\| = \big\|\phi^{-1}(x-y)\big\| \le \frac{1}{m}\,\|x-y\|$, so ϕ^{-1} is Lipschitz continuous.

5.39 Corollary: *When U and V are finite-dimensional normed linear spaces, every linear map $F : U \to V$ is Lipschitz continuous.*

Proof: Let U and V be finite-dimensional vector spaces and let $F : U \to V$ be linear. Let $\{u_1, \cdots, u_n\}$ and $\{v_1, \cdots, v_m\}$ be bases for U and V, and let $\phi : \mathbb{R}^n \to U$ and $\psi : \mathbb{R}^m \to V$ be the vector space isomorphisms given by $\phi(t) = \sum_{k=1}^{n} t_k u_k$ and $\psi(s) = \sum_{k=1}^{m} s_k v_k$. Since ψ^{-1} and ϕ are both linear, the composite $G = \psi^{-1} F \phi : \mathbb{R}^n \to \mathbb{R}^m$ is linear, hence continuous (linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, using the standard metric, are continuous). By the above theorem, we know that ψ and ϕ^{-1} are continuous, and so the composite map $F = \psi\, G \phi^{-1}$ is continuous, hence also Lipschitz continuous, by Theorem 5.35.

5.40 Corollary: *Any two norms on a finite-dimensional vector space U induce the same topology on U.*

Proof: Let U have two norms $\|\ \|_1$ and $\|\ \|_2$, inducing two metrics d_1 and d_2, determining two topologies on U. Let $I : (U, d_1) \to (U, d_2)$ be the identity map (given by $I(x) = x$), and let $J = I^{-1} : (U, d_2) \to (U, d_1)$ (so J is also the identity map). By the above corollary, I and J are continuous. Let $A \subseteq U$. If A is open in (U, d_1) then, since J is continuous, $J^{-1}(A)$ is open in (U, d_2), but $J^{-1}(A) = I(A) = A$ and so A is open in (U, d_2). Similarly, if A is open in (U, d_2) then $A = J(A) = I^{-1}(A)$ is open in (U, d_1).

5.41 Corollary: *Let U be a finite-dimensional vector space. Let $\|\ \|_1$ and $\|\ \|_2$ be two norms on U inducing the two metric d_1 and d_2 on U. Let $(x_n)_{n\geq 1}$ be a sequence in U, and let $a \in U$. Then $x_n \to a$ in (U, d_1) if and only if $x_n \to a$ in (U, d_2).*

Proof: Let $I : (U, d_1) \to (U, d_2)$ be the identity map (given by $I(x) = x$). By Corollary 5.39, I is Lipschitz continuous. Let $\ell \geq 0$ be a Lipschitz constant for I. Suppose that $x_n \to a$ in (U, d_1). Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ such that when $n \geq m$ we have $d_1(x_n, a) < \frac{\epsilon}{\ell+1}$. Then when $n \geq m$ we have $d_2(x_n, a) = d_2\big(I(x_n), I(a)\big) \leq \ell \cdot d_1(x_n, a) < \ell \cdot \frac{\epsilon}{\ell+1} < \epsilon$. Thus $x_n \to a$ in (U, d_2). Similarly, since the identity map $J : (U, d_2) \to (U, d_1)$ is Lipschitz continuous, it follows that if $x_n toa$ in (U, d_2) then $x_n \to a$ in (U, d_1). We remark that I and J might have different Lipschits constants (even though I and J are both the identity map from U to itself).