Chapter 4. Metric Spaces

Introduction

In multivariable calculus, we study vector-valued sequences, that is sequences in $\mathbb{R}^n$, and vector-valued functions of several variables, that is functions $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$. In linear algebra, we study inner product spaces. An n-dimensional inner product space U can be given an orthonormal basis $\{u_1, u_2, \cdots, u_n\}$ (using the Gram-Schmidt Procedure) and then we can identify it with Euclidean space $\mathbb{R}^n$ $\big($using the map $\phi : \mathbb{R}^n \to U$ given by $\phi(t) = \sum_{k=1}^{n} t_k u_k\big)$.

In real analysis, we combine ideas from linear algebra and multivariable calculus. For example, we study sequences in U and functions $f : U \to V$, where U and V are inner product spaces. We are interested not only in n-dimensional inner product spaces (which, as mentioned above, can be identified with $\mathbb{R}^n$), but also in infinite-dimensional vector spaces, such as the vector space of all sequences in $\mathbb{R}$, and the vector space of all functions $f : [a, b] \to \mathbb{R}$.

Recall, from linear algebra, that given an inner product on a vector space U we can define a norm on U by $\|u\| = \sqrt{\langle u, u\rangle}$ and we can define distance between two elements in U by $d(u, v) = \|u - v\|$. In real analysis, we sometimes meet vector spaces which do not have an inner product, but they do have a norm. A vector space with a norm (but not necessarily with an inner product) is called a normed linear space, and we shall define these in this chapter. Also, we sometimes meet spaces (such as arbitrary subsets of inner product spaces) which are not vector spaces but which do have a well-defined notion of distance. Such spaces are called metric spaces, and we shall define these in this chapter.

In multivariable calculus, we often define what it means for a subset $A \subseteq \mathbb{R}^n$ to be open or closed. Intuitively, a subset $A \subseteq \mathbb{R}^2$ is open when it does not include its boundary points (we draw the boundary as a dotted line) and a closed subset does include its boundary points (and we draw the boundary as a solid line). In this chapter, we shall define what it means for a subset $A \subseteq X$ to be open or closed when X is a metric space (including the case that X is an inner product space or a normed linear space), and we shall define exactly what the boundary of such a set is.

In the following chapter (Chapter 5) we shall study sequences in X and functions $f : X \to Y$, where X and Y are metric spaces. We shall study convergence of sequences and limits of functions and continuity of functions (but we shall not deal with differentiation or integration of such functions). Some of the definitions and theorems from calculus can be generalized to apply in this more general setting.

Inner Product Spaces

4.1 Definition: Let U be a vector space over $\mathbb{R}$. An **inner product** on U is a function $\langle\ ,\ \rangle : U\times U\to\mathbb{R}$ (meaning that if $u,v\in U$ then $\langle u,v\rangle\in\mathbb{R}$) such that for all $u,v,w\in U$ and all $t\in\mathbb{R}$ we have

(1) (Bilinearity) $\langle u+v,w\rangle=\langle u,w\rangle+\langle v,w\rangle\ ,\ \langle tu,v\rangle=t\,\langle u,v\rangle$,
$\langle u,v+w\rangle=\langle u,v\rangle+\langle u,w\rangle\ ,\ \langle u,tv\rangle=t\,\langle u,v\rangle$,
(2) (Symmetry) $\langle u,v\rangle=\langle v,u\rangle$, and
(3) (Positive Definiteness) $\langle u,u\rangle\geq 0$ with $\langle u,u\rangle=0\iff u=0$.

For $u,v\in U$, $\langle u,v\rangle$ is called the **inner product** of u with v. We say that u and v are **orthogonal** when $\langle u,v\rangle=0$. An **inner product space** is a vector space (over $\mathbb{R}$) equipped with an inner product. Given two inner product spaces U and V over $\mathbb{R}$, a linear map $L:U\to V$ is called a **homomorphism** of inner product spaces (or we say that L **preserves inner product**) when $\big\langle L(x),L(y)\big\rangle=\langle x,y\rangle$ for all $x,y\in U$. A bijective homomorphism is called an **isomorphism**.

4.2 Example: The **standard inner product** on $\mathbb{R}^n$ is given by

$$\langle u,v\rangle=\sum_{k=1}^{n}u_kv_k.$$

4.3 Example: We write $\mathbb{R}^\omega$ to denote the **space of sequences** in $\mathbb{R}$, and we write $\mathbb{R}^\infty$ to denote the **space of eventually zero sequences** in $\mathbb{R}$, that is

$$\begin{aligned}\mathbb{R}^\omega&=\big\{u=(u_1,u_2,u_3,\cdots)\,\big|\text{ each }u_k\in\mathbb{R}\big\}\\ \mathbb{R}^\infty&=\big\{u\in\mathbb{R}^\omega\,|\,\exists n\in\mathbb{Z}^+\ \forall k\geq n\ \ u_k=0\big\}.\end{aligned}$$

Recall that $\mathbb{R}^\infty$ is an infinite-dimensional vector space with standard basis $\{e_1,e_2,e_3,\cdots\}$ where $e_1=(1,0,0,\cdots)$, $e_2=(0,1,0,\cdots)$ and so on. Note that $\{e_1,e_2,e_3,\cdots\}$ spans $\mathbb{R}^\infty$ (and *not* all of $\mathbb{R}^\omega$) because linear combinations are given by *finite* sums (not by infinite series). The **standard inner product** on $\mathbb{R}^\infty$ is given by

$$\langle u,v\rangle=\sum_{k=1}^{\infty}u_kv_k\,.$$

Note that the sum here does make sense because only finitely many of the terms are nonzero (we cannot use the same formula to give an inner product on $\mathbb{R}^\omega$).

4.4 Example: For $a,b\in\mathbb{R}$ with $a\leq b$, we write

$$\begin{aligned}\mathcal{B}[a,b]&=\mathcal{B}\big([a,b],\mathbb{R}\big)=\big\{f:[a,b]\to\mathbb{R}\,\big|\,f\text{ is bounded}\big\}\\ \mathcal{C}[a,b]&=\mathcal{C}\big([a,b],\mathbb{R}\big)=\big\{f:[a,b]\to\mathbb{R}\big|f\text{ is continuous}\big\}.\end{aligned}$$

The **standard inner product** on $\mathcal{C}[a,b]$ is given by

$$\langle f,g\rangle=\int_a^b f\,g=\int_a^b f(x)g(x)\,dx.$$

Note that this is positive definite because $\langle f,f\rangle=\int_a^b f(x)^2dx\geq 0$ and if $\int_a^b f(x)^2dx=0$ the we must have $f(x)=0$ for all $x\in[a,b]$, using the fact that if g is non-negative and continuous on $[a,b]$ with $\int_a^b g(x)\,dx=0$, then we must have $g(x)=0$ for all $x\in[a,b]$ (we leave the proof of this fact as an exercise).

4.5 Theorem: *Let U be an inner product space and let $u, v \in U$. If $\langle u, x\rangle = \langle v, x\rangle$ for all $x \in U$, then $u = v$.*

Proof: Suppose that $\langle u, x\rangle = \langle v, x\rangle$ for all $x \in U$. Then $\langle u - v, x\rangle = \langle u, x\rangle - \langle v, x\rangle = 0$ for all $x \in U$. In particular, taking $x = u - v$ we have $\langle u - v, u - v\rangle = 0$ so that $u = v$, by positive definiteness.

4.6 Definition: Let U be an inner product space. For $u \in U$, we define the **norm** (or **length**) of u to be

$$\|u\| = \sqrt{\langle u, u\rangle}.$$

4.7 Theorem: *(Basic Properties of Inner Product and Norm) Let U be an inner product space. For $u, v \in U$ and $t \in \mathbb{R}$ we have*

(1) (Scaling) $\|tu\| = |t|\,\|u\|$,
(2) (Positive Definiteness) $\|u\| \geq 0$ *with* $\|u\| = 0 \iff u = 0$,
(3) $\|u \pm v\|^2 = \|u\|^2 \pm 2\,\langle u, v\rangle + \|v\|^2$,
(4) (Pythagoras' Theorem) $\langle u, v\rangle = 0 \iff \|u + v\|^2 = \|u\|^2 + \|v\|^2$,
(5) (Parallelogram Law) $\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$,
(6) (Polarization Identity) $\langle u, v\rangle = \frac{1}{4}\big(\|u + v\|^2 - \|u - v\|^2\big)$,
(7) (The Cauchy-Schwarz Inequality) $\big|\langle u, v\rangle\big| \leq \|u\|\,\|v\|$ *with* $|\langle u, v\rangle| = \|u\|\,\|v\|$ *if and only if* $\{u, v\}$ *is linearly dependent, and*
(8) (The Triangle Inequality) $\big|\|u\| - \|v\|\big| \leq \|u + v\| \leq \|u\| + \|v\|$.

Proof: You will have already seen a proof in a linear algebra course, but let us remind you of some of the proofs. The first 6 parts are all easy to prove. To prove Part 7, suppose first that $\{u, v\}$ is linearly dependent. Then one of u and v is a multiple of the other, say $v = tu$ with $t \in \mathbb{R}$. Then we have $|\langle u, v\rangle| = |\langle u, tu\rangle| = \big|t\langle u, u\rangle\big| = |t|\,\|u\|^2 = \|u\|\,\|tu\| = \|u\|\,\|v\|$. Next suppose that $\{u, v\}$ is linearly independent. Then $1 \cdot v + t \cdot u \neq 0$ for all $t \in \mathbb{R}$, so in particular $v - \frac{\langle u,v\rangle}{\|u\|^2}\,u \neq 0$. Thus we have

$$\begin{aligned} 0 &< \big\|v - \tfrac{\langle u,v\rangle}{\|u\|^2}\,u\big\|^2 = \|v\|^2 - 2\big\langle v, \tfrac{\langle u,v\rangle}{\|u\|^2}\,u\big\rangle + \big\|\tfrac{\langle u,v\rangle}{\|u\|^2}\,u\big\|^2 \\ &= \|v\|^2 - 2\,\tfrac{\langle u,v\rangle^2}{\|u\|^2} + \tfrac{\langle u,v\rangle^2}{\|u\|^2} = \|v\|^2 - \tfrac{\langle u,v\rangle^2}{\|u\|^2} \end{aligned}$$

so that $\frac{\langle u,v\rangle^2}{\|u\|^2} < \|v\|^2$ and hence $|\langle u, v\rangle| \leq \|u\|\,\|v\|$. This proves Part 7.

Using Parts 3 and 7, we have

$$\begin{aligned} \|u + v\|^2 &= \|u\|^2 + 2\,\langle u, v\rangle + \|v\|^2 \leq \|u\|^2 + 2|\langle u, v\rangle| + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\|\,\|v\| + \|v\|^2 = \big(\|u\| + \|v\|\big)^2. \end{aligned}$$

Taking the square root on both sides gives $\|u + v\| \leq \|u\| + \|v\|$. Finally note that $\|u\| = \|(u+v) - v\| \leq \|u+v\| + \|-v\| = \|u+v\| + \|v\|$ so that we have $\|u\| - \|v\| \leq \|u+v\|$, and similarly $\|v\| - \|u\| \leq \|u + v\|$, hence $\big|\|u\| - \|v\|\big| \leq \|u + v\|$. This proves Part 8.

Normed Linear Spaces

4.8 Definition: Let U be a vector space over $\mathbb{R}$. A **norm** on U is a function $\| \ \| : U \to \mathbb{R}$ (meaning that if $u \in U$ then $\|u\| \in \mathbb{R}$) such that for all $u, v \in U$ and all $t \in \mathbb{R}$ we have
(1) (Scaling) $\|tu\| = |t| \, \|u\|$,
(2) (Positive Definiteness) $\|u\| \geq 0$ with $\|u\| = 0 \iff u = 0$, and
(3) (Triangle Inequality) $\|u + v\| \leq \|u\| + \|v\|$.

For $u \in U$, the real number $\|u\|$ is called the **norm** (or **length**) of u, and we say that u is a **unit vector** when $\|u\| = 1$. A **normed linear space** (over $\mathbb{R}$) is a vector space equipped with a norm. Given two normed linear spaces U and V over $\mathbb{R}$, a linear map $L : U \to V$ is called a **homomorphism** of normed linear spaces (or we say that L **preserves norm**) when $\big\|L(x)\big\| = \|x\|$ for all $x \in U$. A bijective homomorphism is called an **isomorphism**.

4.9 Example: The standard inner product on $\mathbb{R}^n$ induces the **standard norm** on $\mathbb{R}^n$, which is also called the **2-norm** on $\mathbb{R}^n$, given by

$$\|u\|_2 = \|u\| = \sqrt{\langle u, u \rangle} = \Big(\sum_{k=1}^{n} |u_k|^2 \Big)^{1/2}.$$

We also define the **1-norm** and the **supremum norm** (also called the **infinity norm**) on $\mathbb{R}^n$ by

$$\begin{aligned} \|u\|_1 &= \sum_{k=1}^{n} |u_k| \,, \\ \|u\|_\infty &= \max \big\{ |u_1|, |u_2|, \cdots, |u_n| \big\}. \end{aligned}$$

4.10 Example: The standard inner product on $\mathbb{R}^\infty$ induces the **standard norm**, also called the **2-norm**, on $\mathbb{R}^\infty$ given by

$$\|u\|_2 = \sqrt{\langle u, u \rangle} = \Big(\sum_{k=1}^{\infty} |u_k|^2 \Big)^{1/2}.$$

We also define the **1-norm** and the **supremum norm** (also called the **infinity norm**) on $\mathbb{R}^\infty$ by

$$\begin{aligned} \|u\|_1 &= \sum_{k=1}^{\infty} |u_k| \,, \\ \|u\|_\infty &= \sup \big\{ |u_k| \,\big|\, k \in \mathbb{Z}^+ \big\} = \max \big\{ |u_k| \,\big|\, k \in \mathbb{Z}^+ \big\}. \end{aligned}$$

4.11 Definition: For $u \in \mathbb{R}^w$, we define the **1-norm** of u, the **2-norm** of u, and the **supremum norm** (or **infinity norm**) of u to be the extended real numbers

$$\|u\|_1 = \sum_{k=1}^{\infty} |u_k| \ , \ \|u\|_2 = \Big(\sum_{k=1}^{\infty} |u_k|^2 \Big)^{1/2} \ \text{ and } \ \|u\|_\infty = \sup \big\{ |u_k| \,\big|\, k \in \mathbb{Z}^+ \big\}$$

Note that these can be infinite, with $\|u\|_\infty = \infty$ in the case that $\big\{ |u_k| \,\big|\, k \in \mathbb{Z}^+ \big\}$ is not bounded above (by a real number). Define

$$\begin{aligned} \ell_1 = \ell_1(\mathbb{R}) &= \big\{ u \in \mathbb{R}^\omega \,\big|\, \|u\|_1 < \infty \big\}, \\ \ell_2 = \ell_2(\mathbb{R}) &= \big\{ u \in \mathbb{R}^\omega \,\big|\, \|u\|_2 < \infty \big\}, \\ \ell_\infty = \ell_\infty(\mathbb{R}) &= \big\{ u \in \mathbb{R}^\omega \,\big|\, \|u\|_\infty < \infty \big\}. \end{aligned}$$

4.12 Example: For the sequence $(u_k)_{k\geq 1}$ in $\mathbb{R}$ given by $u_k = \frac{1}{2^k}$, we have

$$\|u\|_1 = \sum_{k=1}^{\infty} \tfrac{1}{2^k} = 1 \ , \ \|u\|_2 = \Big(\sum_{k=1}^{\infty} \tfrac{1}{4^k}\Big)^{1/2} = \tfrac{1}{\sqrt{3}} \ , \text{ and } \ \|u\|_\infty = |u_1| = \tfrac{1}{2}.$$

For the sequence $(v_k)_{k\geq 1}$ given by $v_k = \frac{1}{k}$, we have

$$\|v\|_1 = \sum_{k=1}^{\infty} \tfrac{1}{k} = \infty \ , \ \|v\|_2 = \Big(\sum_{k=1}^{\infty} \tfrac{1}{k^2}\Big)^{1/2} < \infty \ , \text{ and } \ \|v\|_\infty = |u_1| = 1$$

$\big($in fact $\|v\|_2 = \frac{\pi}{\sqrt{6}}\big)$. For the sequence $(w_k)_{k\geq 1}$ given by $w_k = \frac{1}{\sqrt{k}}$ we have

$$\|w\|_1 = \sum_{k=1}^{\infty} \tfrac{1}{\sqrt{k}} = \infty \ , \ \|w\|_2 = \Big(\sum_{k=1}^{\infty} \tfrac{1}{k}\Big)^{1/2} = \infty \ , \text{ and } \ \|w\|_\infty = |w|_1 = 1.$$

4.13 Theorem: *We have* $\mathbb{R}^\infty \subseteq \ell_1 \subseteq \ell_2 \subseteq \ell_\infty \subseteq \mathbb{R}^\omega$.

Proof: If $u \in \mathbb{R}^\infty$ then $\|u\|_1 = \sum\limits_{k=1}^{\infty} |u_k| < \infty$ (because only finitely many of the terms are nonzero) and so $u \in \ell_1$. Thus we have $\mathbb{R}^\infty \subseteq \ell_1$.

Suppose that $u \in \ell_1$. Since $\|u\|_1 = \sum |u_k| < \infty$, we know that $|u_k| \to 0$ (by the Divergence Test from calculus) so we can choose $m \in \mathbb{Z}^+$ such that when $k \geq m$ we have $|a_k| \leq 1$. Then for $k \geq m$ we have $|a_k|^2 \leq |a_k|$. Since $\sum |a_k|$ converges and $|a_k|^2 \leq |a_k|$ for $k \geq m$, it follows that $\sum |a_k|^2$ converges by the Comparison Test (from calculus). Thus $\|u\|_2 = \Big(\sum\limits_{k=1}^{\infty} |a_k|^2\Big)^{1/2} < \infty$ and so $u \in \ell_2$. Thus we have $\ell_1(\mathbb{R}) \subseteq \ell_2$.

Suppose $u \in \ell_2$. Since ${\|u\|_2}^2 = \sum\limits_{k=1}^{\infty} |a_k|^2 < \infty$ we have $|a_k|^2 \to 0$ (by the Divergence Test) hence also $|a_k| \to 0$. Choose $m \in \mathbb{Z}^+$ such that when $k \geq m$ we have $|a_k| \leq 1$. Then the set $\big\{|a_k| \,\big|\, k \in \mathbb{Z}^+\big\}$ is bounded above by $M = \max\big\{|a_1|, |a_2|, \cdots, |a_{m-1}|, 1\big\}$, and so we have $\|u\|_\infty \leq M$, and hence $u \in \ell_\infty$. Thus we have $\ell_2 \subseteq \ell_\infty$.

Finally note that $\ell_\infty \subseteq \mathbb{R}^\omega$, by definition.

4.14 Theorem:

(1) The space ℓ_2 is an inner product space with inner product defined by

$$\langle u, v\rangle = \sum_{k=1}^{\infty} u_k v_k.$$

(2) For $p = 1, 2, \infty$, the space ℓ_p is a normed linear space with norm given by $\|u\|_p$.

Proof: To prove Part 1, we need to show that if $u, v \in \ell_2$, then $\sum\limits_{k=1}^{\infty} u_k v_k$ converges so the inner product is well defined. Let $u, v \in \ell_2$. For $n \in \mathbb{Z}^+$, let $x = (|u_1|, |u_2|, \cdots, |u_n|) \in \mathbb{R}^n$ and $y = (|v_1|, |v_2|, \cdots, |v_n|) \in \mathbb{R}^n$. Then $\|x\|_2 = \Big(\sum\limits_{k=1}^{n} |u_k|^2\Big)^{1/2} \leq \Big(\sum\limits_{k=1}^{\infty} |u_k|^2\Big)^{1/2} = \|u\|_2$ and similarly $\|y\|_2 \leq \|v\|_2$. By applying the Cauchy-Schwarz Inequality in $\mathbb{R}^n$ we have $\sum\limits_{k=1}^{n} |u_k v_k| = \big|\langle x, y\rangle\big| \leq \|x\|_2 \, \|y\|_2 \leq \|u\|_2 \|v\|_2$. By the Monotone Convergence Theorem, since $\sum\limits_{k=1}^{n} |u_k v_k| \leq \|u\|_2 \|v\|_2$ for every $n \in \mathbb{Z}^+$, it follows that $\sum\limits_{k=1}^{\infty} |u_k v_k| \leq \|u\|_2 \|v\|_2$. Since absolute convergence implies convergence, the series $\sum u_k v_k$ converges, as required.

We leave it as an exercise to verify that the 3 properties which define an inner product (in Definition 3.1) are all satisfied.

Because $\langle u,v\rangle = \sum\limits_{k=1}^{\infty} u_k v_k$ gives a (well-defined, finite-valued) inner product on ℓ_2, it follows (from Theorem 4.7) that this inner product induces a (well-defined, finite-valued) norm on ℓ_2 given by $\|u\| = \sqrt{\langle u,u\rangle} = \Big(\sum\limits_{k=1}^{\infty} |u_k|^2\Big)^{1/2}$. This is the formula we used to define the 2-norm, so the 2-norm is a norm on ℓ_2. To complete the proof of Part 2 of the theorem, it remains to show that $\|u\|_1$ is a norm on ℓ_1, and $\|u\|_\infty$ is a norm on ℓ_∞. We leave this as an exercise $\big($but we remark that that unlike the situation for the inner product $\langle u,v\rangle$, we do not need to verify that $\|u\|_1$ and $\|u\|_\infty$ are finite-valued because this is immediate from the definition of ℓ_1 and $\ell_\infty\big)$.

4.15 Example: For $a,b\in\mathbb{R}$ with $a\le b$, recall that

$$\mathcal{B}[a,b] = \mathcal{B}\big([a,b],\mathbb{R}\big) = \big\{f:[a,b]\to\mathbb{R}\,\big|\,f \text{ is bounded}\big\},$$
$$\mathcal{C}[a,b] = \mathcal{C}\big([a,b],\mathbb{R}\big) = \big\{f:[a,b]\to\mathbb{R}\big|f \text{ is continuous}\big\}.$$

For $f\in\mathcal{C}[a,b]$, we define the **1-norm** and the **2-norm** of f to be

$$\|f\|_1 = \int_a^b |f|\,,$$
$$\|f\|_2 = \left(\int_a^b |f|^2\right)^{1/2}.$$

and for $f\in\mathcal{B}[a,b]$, we define the **supremum norm** (also called the **infinity norm**) of f to be

$$\|f\|_\infty = \sup\Big\{\big|f(x)\big|\,\Big|\,a\le x\le b\Big\}\,.$$

We leave it as an exercise to show that these are indeed norms (in particular, show that the 1-norm is positive-definite). The 2-norm on $\mathcal{C}[a,b]$ is induced by the inner product on $\mathcal{C}[a,b]$ given by

$$\langle f,g\rangle = \int_a^b f\,g = \int_a^b f(x)g(x)\,dx\,.$$

Metric Spaces

4.16 Definition: Let U be a normed linear space. For $u, v \in U$, we define the **distance** between u and v to be
$$d(u,v) = \|v-u\|.$$

4.17 Theorem: *Let U be as normed linear space. For all $u, v, w \in U$,*
(1) (Symmetry) $d(u,v) = d(v,u)$,
(2) (Positive Definiteness) $d(u,v) \geq 0$ with $d(u,v) = 0 \iff u = v$, and
(3) (Triangle Inequality) $d(u,w) \leq d(u,v) + d(v,w)$.

Proof: The proof is left as an easy exercise.

4.18 Definition: Let X be a set. A **metric** on X is a map $d =: X \times X \to \mathbb{R}$ such that for all $a, b, c \in X$ we have
(1) (Symmetry) $d(a,b) = d(b,a)$,
(2) (Positive Definiteness) $d(a,b) \geq 0$ with $d(a,b) = 0 \iff a = b$, and
(3) (Triangle Inequality) $d(a,c) \leq d(a,b) + d(b,c)$.

For $a, b \in X$, $d(a,b)$ is called the **distance** between a and b. A **metric space** is a set X which is equipped with a metric d, and we sometimes denote the metric space by X and sometimes by the pair (X,d). Given two metric spaces (X, d_X) and (Y, d_Y), a map $f: X \to Y$ is called a **homomorphism** of metric spaces (or we say that f is **distance preserving**) when $d_Y\big(f(a), f(b)\big) = d_X(a,b)$ for all $a, b \in X$. A bijective homomorphism is called an **isomorphism** or an **isometry**.

4.19 Note: Every inner product space is also a normed linear space, using the induced norm given by $\|u\| = \sqrt{\langle u, u\rangle}$. Every normed linear space is also a metric space, using the induced metric given by $d(u,v) = \|v-u\|$. If U is an inner product space then every subspace of U is also an inner product space using (the restriction of) the same inner product used in U. If U is a normed linear space then every subspace of U is also a normed linear space using the same norm. If X is a metric space then so is every subset of X using the same metric.

4.20 Example: In $\mathbb{R}^n$ (or in any subset $X \subseteq \mathbb{R}^n$), the **standard metric** (also called the **2-metric**) is given by
$$d(a,b) = d_2(a,b) = \|a-b\|_2 = \Big(\sum_{k=1}^{n} |a_k - b_k|^2\Big)^{1/2}.$$

We also have the **1-metric** and the **supremum metric** (or the **infinity metric**) given by
$$d_1(a,b) = \|a-b\|_1 = \sum_{k=1}^{\infty} |a_k - b_k| \quad \text{and}$$
$$d_\infty(a,b) = \|a-b\|_\infty = \max\big\{|a_k - b_k| \,\big|\, 1 \leq k \leq n\big\}.$$

4.21 Exercise: In $\mathbb{R}^3$, let $u = (1,2,5)$ and $v = (3,5,-1)$. Find $d_1(u,v)$, $d_2(u,v)$ and $d_\infty(u,v)$.

4.22 Example: In ℓ_1 $\big($or in any subset $X \subseteq \ell_1\big)$, we have the **1-metric** given by

$$d_1(a,b) = \|a-b\|_1 = \sum_{k=1}^{\infty} |a_k - b_k| \,.$$

In ℓ_2 $\big($or in any nonempty subset $X \subseteq \ell_2\big)$ we have the **2-metric** given by

$$d_2(a,b) = \|a-b\|_2 = \Big(\sum_{k=1}^{\infty} |a_k - b_k|^2\Big)^{1/2}.$$

In ℓ_∞ $\big($or in any nonempty subset $X \subseteq \ell_\infty\big)$ we have the **supremum metric** (or the **infinity metric**) given by

$$d_\infty(a,b) = \|a-b\|_\infty = \sup\big\{|a_k - b_k| \,\big|\, k \in \mathbb{Z}^+\big\}.$$

Since $\mathbb{R}^\infty \subseteq \ell_1 \subseteq \ell_2 \subseteq \ell_\infty$, we could (if we wanted) also any of the metrics d_p in the space $\mathbb{R}^\infty$ (just as we can use any of the metrics d_p in $\mathbb{R}^n$). We could also use any of the metric d_p in the space ℓ_1, and we could use either of the metrics d_2 or d_∞ in the space ℓ_2.

4.23 Exercise: Let $(u_k)_{k\geq 1}$ and $(v_k)_{k\geq 1}$ be the sequences in ℓ_1 given by $u_k = \frac{1}{2^k}$ and $v_k = \frac{1}{3^k}$. Find $d_1(u,v)$, $d_2(u,v)$ and $d_\infty(u,v)$.

4.24 Example: Let $a, b \in \mathbb{R}$ with $a \leq b$. In $\mathcal{C}[a,b]$ $\big($or in any subset $X \subseteq \mathcal{C}[a,b]\big)$, we have the **1-metric** and the **2-metric**, given by

$$\begin{aligned} d_1(f,g) &= \|f-g\|_1 = \int_a^b |f-g| = \int_a^b \big|f(x)-g(x)\big|\,dx\,, \\ d_2(f,g) &= \|f-g\|_2 = \left(\int_a^b |f-g|^2\right)^{1/2} = \left(\int_a^b \big|f(x)-g(x)\big|^2 dx\right)^{1/2}, \end{aligned}$$

and in $\mathcal{B}[a,b]$ $\big($or in any subset $X \subseteq \mathcal{B}[a,b]\big)$ we have the supremum metric (also called the **infinity metric**) given by

$$d_\infty(f,g) = \|f-g\|_\infty = \max\Big\{\big|f(x)-g(x)\big| \,\Big|\, a \leq x \leq b\Big\}.$$

4.25 Exercise: Define $f, g : [0,1] \to \mathbb{R}$ be $f(x) = x$ and $g(x) = x^2$. Find $d_1(f,g)$, $d_2(f,g)$ and $d_\infty(f,g)$.

4.26 Example: For any nonempty set $X \neq \emptyset$, the **discrete metric** on X is given by $d(x,y) = 1$ for all $x, y \in X$ with $x \neq y$ and $d(x,x) = 0$ for all $x \in X$.

4.27 Remark: There are, in fact, a ridiculously vast number of metrics that one could define on $\mathbb{R}$. For example, if we let $f : \mathbb{R} \to \mathbb{R}$ be any bijective map then we can define a metric on $\mathbb{R}$ by $d(x,y) = |f(x) - f(y)|$. But in this course, we shall usually concern ourselves with the metrics described in the above examples.

Open and Closed Sets in Metric Spaces

4.28 Definition: Let X be a metric space. For $a \in X$ and $0 < r \in \mathbb{R}$, the **open ball**, the **closed ball**, and the (open) **punctured ball** in X centred at a of radius r are defined to be the sets

$$\begin{aligned} B(a,r) = B_X(a,r) &= \{x \in X \,|\, d(x,a) < r\}, \\ \overline{B}(a,r) = \overline{B}_X(a,r) &= \{x \in X \,|\, d(x,a) \leq r\}, \\ B^*(a,r) = B^*_X(a,r) &= \{x \in X \,|\, 0 < d(x,a) < r\}. \end{aligned}$$

When the metric on X is denoted by d_p with $1 \leq p \leq \infty$, we often write $B(a,r)$, $\overline{B}(a,r)$ and $B^*(a,r)$ as $B_p(a,r)$, $\overline{B}_p(a,r)$ and $B^*_p(a,r)$. For $A \subseteq X$, we say that A is **bounded** when $A \subseteq B(a,r)$ for some $a \in X$ and some $0 < r \in \mathbb{R}$.

4.29 Exercise: Draw a picture of the open balls $B_1(0,1)$, $B_2(0,1)$ and $B_\infty(0,1)$ in $\mathbb{R}^2$ (using the metrics d_1, d_2 and d_∞).

4.30 Definition: Let X be a metric space. For $A \subseteq X$, we say that A is **open** (in X) when for every $a \in A$ there exists $r > 0$ such that $B(a,r) \subseteq A$, and we say that A is **closed** (in X) when its complement $A^c = X \setminus A$ is open in X.

4.31 Example: Let X be a metric space and let $a \in X$. Show that $\{a\}$ is closed in X.

Solution: To show that $\{a\}$ is closed, we shall show that $\{a\}^c = X \setminus \{a\}$ is open. Let $b \in X \setminus \{a\}$. Let $r = d(a,b)$ and note that since $b \neq a$ we have $r > 0$. Let $x \in B(b,r)$. Then $d(x,b) < r = d(a,b)$. Since $d(x,b) \neq d(a,b)$ we have $x \neq a$ so that $x \in X \setminus \{a\}$. Thus $B(b,r) \subseteq X \setminus \{a\}$. This proves that $X \setminus \{a\}$ is open, and so $\{a\}$ is closed.

4.32 Example: Let X be a metric space. Show that for $a \in X$ and $0 < r \in \mathbb{R}$, the set $B(a,r)$ is open and the set $\overline{B}(a,r)$ is closed.

Solution: Let $a \in X$ and let $r > 0$. We claim that $B(a,r)$ is open. We need to show that for all $b \in B(a,r)$ there exists $s > 0$ such that $B(b,s) \subseteq B(a,r)$. Let $b \in B(a,r)$ and note that $d(a,b) < r$. Let $s = r - d(a,b)$ and note that $s > 0$. Let $x \in B(b,s)$, so we have $d(x,b) < s$. Then, by the Triangle Inequality, we have

$$d(x,a) \leq d(x,b) + d(b,a) < s + d(a,b) = r$$

and so $x \in B(a,r)$. This shows that $B(b,s) \subseteq B(a,r)$ and hence $B(a,r)$ is open.

Next we claim that $\overline{B}(a,r)$ is closed, that is $\overline{B}(a,r)^c$ is open. Let $b \in \overline{B}(a,r)^c$, that is let $b \in X$ with $b \notin \overline{B}(a,r)$. Since $b \notin \overline{B}(a,r)$ we have $d(a,b) > r$. Let $s = d(a,b) - r > 0$. Let $x \in B(b,s)$ and note that $d(x,b) < s$. Then, by the Triangle Inequality, we have

$$d(a,b) \leq d(a,x) + d(x,b) < d(x,a) + s$$

and so $d(x,a) > d(a,b) - s = r$. Since $d(x,a) > r$ we have $x \notin \overline{B}(a,r)$ and so $x \in \overline{B}(a,r)^c$. This shows that $B(b,s) \subseteq \overline{B}(a,r)^c$ and it follows that $\overline{B}(a,r)^c$ is open and hence that $\overline{B}(a,r)$ is closed.

4.33 Example: In $\mathbb{R}$ (using its standard metric), an open ball is the same thing as a bounded non-degenerate open interval, and a closed ball is the same thing as a bounded non-degenerate closed interval. The unbounded open intervals (a,∞), $(-\infty,b)$ are open, and the unbounded closed intervals $[a,\infty)$ and $(-\infty,b]$ are closed. The degenerate closed intervals $[a,a] = \{a\}$ are closed. The degenerate interval $(a,a) = \emptyset$ and the interval $(-\infty,\infty) = \mathbb{R}$ are both open and closed (see Theorem 4.34 below). The bounded non-degenerate half-open intervals $[a,b)$ and $(a,b]$ are neither open nor closed.

4.34 Remark: It is often fairly difficult to determine whether a given set is open or closed (or neither or both) directly from the definition of open and closed sets. We will be able to do this more easily after we have discussed limits of sequences and continuous functions in the next chapter.

4.35 Theorem: *(Basic Properties of Open Sets) Let X be a metric space.*

(1) The sets $\emptyset$ and X are open in X.
(2) If S is a set of open sets in X then the union $\bigcup S = \bigcup\limits_{U\in S} U$ is open in X.
(3) If S is a finite set of open sets in X then the intersection $\bigcap S = \bigcap\limits_{U\in S} U$ is open in X.

Proof: The empty set is open because any statement of the form "for all $x\in\emptyset$ F" (where F is any statement) is considered to be true (by convention). The set X is open because given $a \in X$ we can choose any value of $r > 0$ and then we have $B(a,r) \subseteq X$ by the definition of $B(a,r)$. This proves Part 1.

To prove Part 2, let S be any set of open sets in X. Let $a \in \bigcup S = \bigcup_{U\in S} U$. Choose an open set $U \in S$ such that $a \in U$. Since U is open we can choose $r > 0$ such that $B(a,r) \subseteq U$. Since $U \in S$ we have $U \subseteq \bigcup S$. Since $B(a,r) \subseteq U$ and $U \subseteq \bigcup S$ we have $B(a,r) \subseteq \bigcup S$. Thus $\bigcup S$ is open, as required.

To prove Part 3, let S be a finite set of open sets in X. If $S = \emptyset$ then we use the convention that $\bigcap S = X$, which is open. Suppose that $S \neq \emptyset$, say $S = \{U_1, U_2, \cdots, U_m\}$ where each U_k is an open set. Let $a \in \bigcap S = \bigcap_{k=1}^m U_k$. For each index k, since $a \in U_k$ we can choose $r_k > 0$ so that $B(a,r_k) \subseteq U_k$. Let $r = \min\{r_1, r_2, \cdots, r_m\}$. Then for each index k we have $B(a,r) \subseteq B(a,r_k) \subseteq U_k$. Since $B(a,r) \subseteq U_k$ for every index k, it follows that $B(a,r) \subseteq \bigcap_{k=1}^m U_k = \bigcap S$. Thus $\bigcap S$ is open, as required.

4.36 Theorem: *(Basic Properties of Closed Sets) Let X be a metric space.*

(1) The sets $\emptyset$ and X are closed in X.
(2) If S is a set of closed sets in X then the intersection $\bigcap S = \bigcap\limits_{K\in S} K$ is closed in X.
(3) If S is a finite set of closed sets in X then the union $\bigcup S = \bigcup\limits_{K\in S} K$ is closed in X.

Proof: This follows from Theorem 4.34, by taking complements using the fact that for a set S of subsets of X we have $\Big(\bigcup\limits_{A\in S} A\Big)^c = \bigcap\limits_{A\in S} A^c$ and $\Big(\bigcap\limits_{A\in S} A\Big)^c = \bigcup\limits_{A\in S} A^c$ (these rules are called DeMorgan's Laws, and you should convince yourself that they are true if you have not seen them).

4.37 Example: When X is a metric space, $a \in X$ and $r > 0$, the punctured ball $B^*(a,r)$ is open (by Part 2 of Theorem 4.34) because $B^*(a,r) = B(a,r) \setminus \{a\} = B(a,r) \cap \{a\}^c$, and the sets $B(a,r)$ and $\{a\}^c$ are both open.

4.38 Example: In $\mathbb{R}$, note that $\bigcap\limits_{n=1}^{\infty} \big(-\frac{1}{n}, 1+\frac{1}{n}\big) = [0,1]$, which is closed and not open, so the intersection of an infinite set of open sets is not always open. Similarly, note that $\bigcup\limits_{n=1}^{\infty} \big[\frac{1}{n}, 1-\frac{1}{n}\big] = (0,1)$, which is open and not closed, so the union of an infinite set of closed sets is not always closed.,

Topological Spaces

4.39 Definition: A **topology** on a set X is a set T of subsets of X such that
(1) $\emptyset \in T$ and $X \in T$,
(2) for every set $S \subseteq T$ we have $\bigcup S \in T$, and
(3) for every finite subset $S \subseteq T$ we have $\bigcap S \in T$.

A **topological space** is a set X with a topology T. When X is a metric space, the set of all open sets in X is a topology on X, which we call the **metric topology** (or the topology **induced** by the metric). When X is any topological space, the sets in the topology T are called the **open sets** in X and their complements are called the **closed sets** in X. When S and T are both topologies on a set X with $S \subseteq T$, we say that the topology T is **finer** than the topology S, and that the topology S is **coarser** than the topology T. When $S \subsetneqq T$ we say that T is **strictly finer** than S and that S is **strictly coarser** than T.

4.40 Example: Show that on $\mathbb{R}^n$, the metrics d_1, d_2 and d_∞ all induce the same topology.

Solution: For $a, x \in \mathbb{R}^n$ we have

$$\max_{1 \leq i \leq n} |x_i - a_i| \leq \Big(\sum_{i=1}^n |x_i - a_i|^2 \Big)^{1/2} \leq \sum_{i=1}^n |x_i - a_i| \leq n \, \max_{i=1}^n |x_i - a_i|$$

and so

$$d_\infty(a,x) \leq d_2(a,x) \leq d_1(a,x) \leq n \, d_\infty(a,x).$$

It follows that for all $a \in \mathbb{R}^n$ and $r > 0$ we have

$$B_\infty(a,r) \supseteq B_2(a,r) \supseteq B_1(a,r) \supseteq B_\infty\big(a, \tfrac{r}{n}\big).$$

Thus for $U \subseteq \mathbb{R}^n$, if U is open in $\mathbb{R}^n$ using d_∞ then it is open using d_2, and if U is open using d_2 then it is open using d_1, and if U is open using d_1 then it is open using d_∞.

4.41 Example: Show that on the space $\mathcal{C}[a,b]$, the topology induced by the metric d_∞ is strictly finer than the topology induced by the metric d_1.

Solution: For $f, g \in \mathcal{C}[a,b]$ we have

$$d_1(f,g) = \int_a^b |f - g| \leq \int_a^b \max_{a \leq x \leq b} \big|f(x) - g(x)\big| = (b-a)\, d_\infty(f,g).$$

It follows that for $f \in \mathcal{C}[a,b]$ and $r > 0$ we have

$$B_\infty(f,r) \subseteq B_1\big(f, (b-a)r\big).$$

Thus for $U \subseteq \mathcal{C}[a,b]$, if U is open using d_1 then U is also open using d_∞, and so the topology induced by the metric d_∞ is finer (or equal to) the topology induced by d_1.

On the other hand, we claim that for $f \in \mathcal{C}[a,b]$ and $r > 0$, the set $B_\infty(f,r)$ is not open in the topology induced by d_1. Fix $g \in B_\infty(f,r)$ and let $s > 0$. Choose a bump function $h \in \mathcal{C}[a,b]$ with $h \geq 0$, $\int_a^b h < s$ and $\max_{a \leq x \leq b} h(x) > 2r$ $\big($for example, choose $c \in (a,b)$ with $c - a < \frac{s}{2r}$ and then define h by $h(x) = 3r\big(1 - \frac{x-a}{c-a}\big)$ for $a \leq x \leq c$ and $h(x) = 0$ for $c \leq x \leq b\big)$. Then we have $g + h \in B_1(g,s)$ but $g + h \notin B_\infty(f,r)$. It follows that $B_\infty(f,r)$ is not open in the topology induced by d_1, as claimed.

4.42 Example: For any set X, the **trivial topology** on X is the the topology in which the only open sets in X are the sets $\emptyset$ and X, and the **discrete topology** on X is the topology in which every subset of X is open. Note that the discrete metric on a nonempty set X induces the discrete topology on X.

Interior and Closure

4.43 Definition: Let X be a metric space (or a topological space) and let $A \subseteq X$. The **interior**, the **closure**, and the **boundary** of A (in X) are the sets

$$\begin{aligned} A^o &= \bigcup \big\{ U \subseteq X \,\big|\, U \text{ is open, and } U \subseteq A \big\}, \\ \overline{A} &= \bigcap \big\{ K \subseteq X \,\big|\, K \text{ is closed and } A \subseteq K \big\} \\ \partial A &= \overline{A} \setminus A^o = \big\{ x \in \overline{A} \,\big|\, x \notin A^o \big\}. \end{aligned}$$

We say that A is **dense** in X when $\overline{A} = X$.

4.44 Theorem: *Let X be a metric space (or a topological space) and let $A \subseteq X$.*

(1) The interior of A is the largest open set which is contained in A. Indeed A^o is open, $A^o \subseteq A$, and and $U \subseteq A^o$ for every open set U with $U \subseteq A$.

(2) The closure of A is the smallest closed set which contains A. Indeed $\overline{A}$ is closed, $A \subseteq \overline{A}$, and and $\overline{A} \subseteq K$ for every closed set K with $A \subseteq K$.

Proof: Let $S = \big\{ U \subseteq X \,\big|\, U \text{ is open, and } U \subseteq A \big\}$. Note that A^o is open because A^o is equal to the union of S, which is a set of open sets. Also note that $A^o \subseteq A$ because A^o is equal to the union of S, which is a set of subsets of A. Finally note that for any open set U with $U \subseteq A$ we have $U \in S$ so that $U \subseteq \bigcup S = A^o$. This completes the proof of Part 1, and the proof of Part 2 is similar.

4.45 Corollary: *Let X be a metric space (or a topological space) and let $A \subseteq X$.*
(1) $(A^o)^o = A^o$ and $\overline{\overline{A}} = \overline{A}$.
(2) A is open if and only if $A = A^o$
(3) A is closed if and only if $A = \overline{A}$.

Proof: The proof is left as an exercise.

Interior Points, Closure Points, Boundary Points, and Limit Points

4.46 Definition: Let X be a metric space and let $A \subseteq X$. An **interior point** of A is a point $a \in A^o$. A **closure point** of A is a point $a \in \overline{A}$. A **boundary point** of A is a point $a \in \partial A$. A **limit point** of A is a point $a \in X$ such that for every open set U in X with $a \in U$ we have $(U \setminus \{a\}) \cap A \neq \emptyset$. The set of limit points of A in X is denoted by A'. An **isolated point** of A is a point $a \in A$ which is not a limit point of A.

4.47 Theorem: *(Charaterization of Interior, Closure, Boundary, and Limit Points) Let X be a metric space, let $A \subseteq X$, and let $a \in X$.*
(1) $a \in A^o$ if and only if $\exists\, r > 0\ B(a,r) \subseteq A$.
(2) $a \in \overline{A}$ if and only if $\forall\, r > 0\ B(a,r) \cap A \neq \emptyset$.
(3) $a \in \partial A$ if and only if $\forall\, r > 0\ \big(B(a,r) \cap A \neq \emptyset$ and $B(a,r) \cap A^c \neq \emptyset\big)$ where $A^c = X \setminus A$.
(4) $a \in A'$ if and only if $\forall\, r > 0\ B^*(a,r) \cap A \neq \emptyset$.

Proof: We prove parts 1 and 3, and leave the proofs of Parts 2 and 4 as exercises. To prove Part 1 note that, from the definition of A^o, we have $a \in A^o$ if and only if there exists an open set U in X with $a \in U \subseteq A$. Suppose that $a \in A^o$. Choose an open set U in X with $a \in U \subseteq A$. Since U is open and $a \in U$, we can choose $r > 0$ such that $B(a,r) \subseteq U$. Then we have $B(a,r) \subseteq U \subseteq A$. This proves that $\exists\, r > 0\ B(a,r) \subseteq A$.

Suppose, conversely, that $\exists\, r > 0\ B(a,r) \subseteq A$. Choose $r > 0$ such that $B(a,r) \subseteq A$, and let $U = B(a,r)$. Then U is an open set in X with $a \in U \subseteq A$, and hence $a \in A^o$. This completes the proof of Part 1.

We leave the proof of Parts 2 and 4 as exercises. Part 3 follows from Parts 1 and 2 because $a \in A^o \iff \exists\, r > 0\ B(a,r) \subseteq A \iff \exists\, r > 0\ B(a,r) \cap A^c = \emptyset$, and so

$$\begin{aligned} a \in \partial A &\iff a \in \overline{A} \setminus A^o \iff \big(a \in \overline{A} \text{ and } a \notin A^o\big) \\ &\iff \forall r > 0\ B(a,r) \cap A \neq \emptyset \text{ and } \forall\, r > 0\ B(a,r) \cap A^c \neq \emptyset \\ &\iff \forall r > 0\ \big(B(a,r) \cap A \neq \emptyset \text{ and } \forall\, r > 0\ B(a,r) \cap A^c\big). \end{aligned}$$

4.48 Theorem: *(Closure and Limit Points) Let X be a metric space and let $A \subseteq X$.*
(1) A is closed if and only if $A' \subseteq A$.
(2) $\overline{A} = A \cup A'$.

Proof: To prove Part 1, note that when $a \notin A$ we have $B(a,r) \cap A = B^*(a,r) \cap A$ and so

$$\begin{aligned} A \text{ is closed} &\iff A^c \text{ is open} \\ &\iff \forall a \in A^c\ \exists r > 0\ B(a,r) \subseteq A^c \\ &\iff \forall a \in \mathbb{R}^n\ \big(a \notin A \Longrightarrow \exists r > 0\ B(a,r) \subseteq A^c \\ &\iff \forall a \in \mathbb{R}^n\ \big(a \notin A \Longrightarrow \exists r > 0\ B(a,r) \cap A = \emptyset\big) \\ &\iff \forall a \in \mathbb{R}^n\ \big(a \notin A \Longrightarrow \exists r > 0\ B^*(a,r) \cap A = \emptyset\big) \\ &\iff \forall a \in \mathbb{R}^n\ \big(\forall r > 0\ B^*(a,r) \cap A \neq \emptyset \Longrightarrow a \in A\big) \\ &\iff \forall a \in \mathbb{R}^n\ \big(a \in A' \Longrightarrow a \in A\big) \\ &\iff A' \subseteq A. \end{aligned}$$

To prove Part 2 we shall prove that $A \cup A'$ is the smallest closed set which contains A. It is clear that $A \cup A'$ contains A. We claim that $A \cup A'$ is closed, that is $(A \cup A')^c$ is open. Let $a \in (A \cup A')^c$, that is let $a \in X$ with $a \notin A$ and $a \notin A'$. Since $a \notin A'$ we can choose $r > 0$ so that $B(a,r) \cap A = \emptyset$. We claim that because $B(a,r) \cap A = \emptyset$ it follows that $B(a,r) \cap A' = \emptyset$. Suppose, for a contradiction, that $B(a,r) \cap A' \neq \emptyset$. Choose $b \in B(a,r) \cap A'$. Since $b \in B(a,r)$ and $B(a,r)$ is open, we can choose $s > 0$ so that $B(b,s) \subseteq B(a,r)$. Since $b \in A'$ it follows that $B(b,s) \cap A \neq \emptyset$. Choose $x \in B(b,s) \cap A$. Then we have $x \in B(b,s) \subseteq B(a,r)$ and $x \in A$ and so $x \in B(a,r) \cap A$, which contradicts the fact that $B(a,r) \cap A = \emptyset$. Thus $B(a,r) \cap A' = \emptyset$, as claimed. Since $B(a,r) \cap A = \emptyset$ and $B(a,r) \cap A' = \emptyset$ it follows that $B(a,r) \cap (A \cup A') = \emptyset$ hence $B(a,r) \subseteq (A \cup A')^c$. Thus proves that $(A \cup A')^c$ is open, and hence $A \cup A'$ is closed.

It remains to show that for every closed set K in X with $A \subseteq K$ we have $A \cup A' \subseteq K$. Let K be a closed set in X with $A \subseteq K$. Note that since $A \subseteq K$ it follows that $A' \subseteq K'$ because if $a \in A'$ then for all $r > 0$ we have $B(a,r) \cap A \neq \emptyset$ hence $B(a,r) \cap K \neq \emptyset$ and so $a \in K'$. Since K is closed we have $K' \subseteq K$ by Part 2. Since $A' \subseteq K'$ and $K' \subseteq K$ we have $A' \subseteq K$. Since $A \subseteq K$ and $A' \subseteq K$ we have $A \cup A' \subseteq K$, as required.

Open and Closed Sets in Subspaces

4.49 Note: Let X be a metric space and let $P \subseteq X$. Note that P is also a metric space using (the restriction of) the metric used in X. For $a \in P$ and $0 < r \in \mathbb{R}$, note that the open and closed balls in P, centred at a and of radius r, are related to the open and closed balls in X by

$$B_P(a,r) = \big\{x \in P \,\big|\, d(x,a) < r\big\} = B_X(a,r) \cap P,$$
$$\overline{B}_P(a,r) = \big\{x \in P \,\big|\, d(x,a) \le r\big\} = \overline{B}_X(a,r) \cap P.$$

4.50 Theorem: *Let X be a metric space and let $A \subseteq P \subseteq X$.*

(1) A is open in P if and only if there exists an open set U in X such that $A = U \cap P$.
(2) A is closed in P if and only if there exists a closed set K in X such that $A = K \cap P$.

Proof: To prove Part 1, suppose first that A is open in P. For each $a \in A$, choose $r_a > 0$ so that $B_P(a,r_a) \subseteq A$, that is $B_X(a,r_a) \cap P \subseteq A$, and let $U = \bigcup_{a \in A} B_X(a,r_a)$. Since U is equal to the union of a set of open sets in X, it follows that U is open in X. Note that $A \subseteq U \cap P$ and, since $B_X(a,r_a) \cap P \subseteq A$ for every $a \in A$, we also have $U \cap P = \Big(\bigcup_{a \in U} B_X(a,r_a)\Big) \cap P = \bigcup_{a \in A} \big(B_X(a,r_a) \cap P\big) \subseteq A$. Thus $A = U \cap P$, as required.

Suppose, conversely, that $A = U \cap P$ with U open in X. Let $a \in A$. Since we have $a \in A = U \cap P$, we also have $a \in U$. Since $a \in U$ and U is open in X we can choose $r > 0$ so that $B_X(a,r) \subseteq U$. Since $B_X(a,r) \subseteq U$ and $U \cap P = A$ we have $B_P(a,r) = B_X(a,r) \cap P \subseteq U \cap P = A$. Thus A is open, as required.

To prove Part 2, suppose first that A is closed in P. Let B be the complement of A in P, that is $B = P \setminus A$. Then B is open in P. Choose an open set U in X such that $B = U \cap P$. Let K be the complement of U in X, that is $K = X \setminus U$. Then $A = K \cap P$ since for $x \in X$ we have $x \in A \iff \big(x \in P \text{ and } x \notin B\big) \iff \big(x \in P \text{ and } x \notin U \cap P\big) \iff \big(x \in P \text{ and } x \notin U\big) \iff \big(x \in P \text{ and } x \in K\big) \iff x \in K \cap P$.

Suppose, conversely, that K is a closed set in P with $A = K \cap P$. Let B be the complement of A in P, that is $B = P \setminus A$, and let U be the complement of K in P, that is $U = P \setminus K$, and note that U is open in P. Then we have $B = U \cap P$ since for $x \in P$ we have $x \in B \iff \big(x \in P \text{ and } x \notin A\big) \iff \big(x \in P \text{ and } x \notin K \cap P\big) \iff \big(x \in P \text{ and } x \notin K\big) \iff \big(x \in P \text{ and } x \in U\big) \iff x \in U \cap P$. Since U is open in P and $B = U \cap P$ we know that B is open in P. Since B is open in P, its complement $A = P \setminus B$ is closed in P.

4.51 Remark: Let X be a topological space and let $P \subseteq X$. Verify, as an exercise, that we can use the topology on X to define a topology on P as follows. Given a set $A \subseteq P$, we define A to be **open** in P when $A = U \cap P$ for some open set U in X. The resulting topology on P is called the **subspace topology**.