Chapter 3. Sequences and Series of Functions

Pointwise Convergence

3.1 Definition: Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and for each integer $n \geq p$ let $f_n : A \to \mathbb{R}$. We say that the sequence of functions $(f_n)_{n \geq p}$ **converges pointwise** to f on A, and we write $f_n \to f$ pointwise on A, when $\lim\limits_{n \to \infty} f_n(x) = f(x)$ for all $x \in A$, that is when for all $x \in A$ and for all $\epsilon > 0$ there exists $m \geq p$ such that for all integers n we have

$$n \geq m \Longrightarrow |f_n(x) - f(x)| < \epsilon\,.$$

3.2 Note: By the Cauchy Criterion for convergence, the sequence $(f_n)_{n \geq p}$ converges pointwise to some function $f(x)$ on A if and only if for all $x \in A$ and for all $\epsilon > 0$ there exists $m \geq p$ such that for all integers k, ℓ we have

$$k, \ell \geq m \Longrightarrow \big|f_k(x) - f_\ell(x)\big| < \epsilon\,.$$

3.3 Example: Find an example of a sequence of functions $(f_n)_{n \geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is continuous but f is not.

Solution: Let $f_n(x) = x^n$. Then $\lim\limits_{n \to \infty} f_n(x) = \begin{cases} 0 \text{ if } x \neq 1 \\ 1 \text{ if } x = 1\,. \end{cases}$

3.4 Example: Find an example of a sequence of functions $(f_n)_{n \geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is differentiable and f is differentiable, but $\lim\limits_{n \to \infty} {f_n}' \neq f'$.

Solution: Let $f_n(x) = \frac{1}{n}\tan^{-1}(nx)$. Then $\lim\limits_{n \to \infty} f_n(x) = 0$, and ${f_n}'(x) = \dfrac{1}{1 + (nx)^2}$ so $\lim\limits_{n \to \infty} {f_n}'(x) = \begin{cases} 0 \text{ if } x \neq 0 \\ 1 \text{ if } x = 0\,. \end{cases}$

3.5 Example: Find an example of a sequence of functions $(f_n)_{n \geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is integrable but f is not.

Solution: We have $\mathbb{Q} \cap [0,1] = \{a_1, a_2, a_3, \cdots\}$ where

$$(a_n)_{n \geq 1} = \left(\tfrac{0}{1}, \tfrac{1}{1}, \tfrac{0}{2}, \tfrac{1}{2}, \tfrac{2}{2}, \tfrac{0}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{3}, \tfrac{0}{4}, \cdots, \tfrac{4}{4}, \cdots\right).$$

(as an exercise, you can check that $a_n = \frac{k}{\ell}$ where $\ell = \left\lceil \frac{-3 + \sqrt{9 - 8n}}{2} \right\rceil$ and $k = n - \frac{\ell^2 + \ell}{2}$).
For $x \in [0,1]$, let $f_n(x) = \begin{cases} 0 \text{ if } x \notin \{a_1, a_2, \cdots, a_n\} \\ 1 \text{ if } x \in \{a_1, a_2, \cdots, a_n\}\,. \end{cases}$ Then $\lim\limits_{n \to \infty} f_n(x) = \begin{cases} 0 \text{ if } x \notin \mathbb{Q} \\ 1 \text{ if } x \in \mathbb{Q}\,. \end{cases}$

3.6 Example: Find an example of a sequence of functions $(f_n)_{n \geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is integrable and f is integrable but

$$\lim_{n \to \infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx\,.$$

Solution: Let $f_1(x) = \begin{cases} 48\left(x - \frac{1}{2}\right)(1 - x) \text{ if } \frac{1}{2} \leq x \leq 1, \\ 0 \text{ otherwise}\,. \end{cases}$ For $n \geq 1$ let $f_n(x) = n f_1(nx)$.

Then each f_n is continuous with $\displaystyle\int_0^1 f_n(x)\,dx = 1$, and $\lim\limits_{n \to \infty} f_n(x) = 0$ for all x.

Uniform Convergence

3.7 Definition: Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and for each integer $n \geq p$ let $f_n : A \to \mathbb{R}$. We say that the sequence of functions $(f_n)_{n \geq p}$ **converges uniformly** to f on A, and we write $f_n \to f$ uniformly on A, when for all $\epsilon > 0$ there exists $m \in \mathbb{Z}_{\geq p}$ such that for all $x \in A$ and for all integers $n \in \mathbb{Z}_{\geq p}$ we have

$$n \geq m \Longrightarrow \left|f_n(x) - f(x)\right| < \epsilon\,.$$

3.8 Theorem: *(Cauchy Criterion for Uniform Convergence of Sequences of Functions) Let $(f_n)_{n \geq p}$ be a sequence of functions on $A \subseteq \mathbb{R}$. Then (f_n) converges uniformly (to some function f) on A if and only if for all $\epsilon > 0$ there exists $m \in \mathbb{Z}_{\geq p}$ such that for all $x \in A$ and for all integers $k, \ell \in \mathbb{Z}_{\geq p}$ we have*

$$k, \ell \geq m \Longrightarrow \left|f_k(x) - f_\ell(x)\right| < \epsilon\,.$$

Proof: Suppose that (f_n) converges uniformly to f on A. Let $\epsilon > 0$. Choose m so that for all $x \in A$ we have $n \geq m \Longrightarrow \left|f_n(x) - f(x)\right| < \frac{\epsilon}{2}$. Then for $k, \ell \geq m$ we have $\left|f_k(x) - f(x)\right| < \frac{\epsilon}{2}$ and $\left|f_\ell(x) - f(x)\right| < \frac{\epsilon}{2}$ and so

$$\left|f_k(x) - f_\ell(x)\right| \leq \left|f_k(x) - f(x)\right| + \left|f_\ell(x) - f(x)\right| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,.$$

Conversely, suppose that (f_n) satisfies the Cauchy Criterion for uniform convergence, that is for all $\epsilon > 0$ there exists m such that for all $x \in A$ and all integers n, ℓ we have

$$n, \ell \geq m \Longrightarrow \left|f_n(x) - f_\ell(x)\right| < \epsilon\,.$$

For each fixed $x \in A$, $(f_n(x))$ is a Cauchy sequence, so $(f_n(x))$ converges, and we can define $f(x)$ by

$$f(x) = \lim_{n \to \infty} f_n(x)\,.$$

We know that $f_n \to f$ pointwise on A, but we must show that $f_n \to f$ uniformly on A. Let $\epsilon > 0$. Choose m so that for all $x \in A$ and for all integers n, ℓ we have

$$n, \ell \geq m \Longrightarrow \left|f_n(x) - f_\ell(x)\right| < \tfrac{\epsilon}{2}\,.$$

Let $x \in A$. Since $\lim\limits_{\ell \to \infty} f_\ell(x) = f(x)$, we can choose $\ell \geq m$ so that $\left|f_\ell(x) - f(x)\right| < \frac{\epsilon}{2}$. Then for $n \geq m$ we have

$$\left|f_n(x) - f(x)\right| \leq \left|f_n(x) - f_\ell(x)\right| + \left|f_\ell(x) - f(x)\right| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,.$$

3.9 Theorem: *(Uniform Convergence, Limits and Continuity) Suppose that $f_n \to f$ uniformly on A. Let x be a limit point of A. If $\lim\limits_{y \to x} f_n(y)$ exists for each n, then*

$$\lim_{y \to x} \lim_{n \to \infty} f_n(y) = \lim_{n \to \infty} \lim_{y \to x} f_n(y)\,.$$

In particular, if each f_n is continuous in A, then so is f.

Proof: Suppose that $\lim\limits_{y \to x} f_n(y)$ exists for all n. Let $b_n = \lim\limits_{y \to x} f_n(y)$. We must show that $\lim\limits_{y \to x} f(y) = \lim\limits_{n \to \infty} b_n$. We claim first that (b_n) converges. Let $\epsilon > 0$. Since (f_n) converges uniformly, it satisfies the Cauchy criterion, so we can choose m so that $k, \ell \geq m \Longrightarrow \left|f_k(y) - f_\ell(y)\right| < \frac{\epsilon}{3}$ for all $y \in A$. Let $k, \ell \geq m$. Since $f_k(y) \to b_k$ and $f_\ell(y) \to b_\ell$ as $y \to x$, we can choose $y \in A$ so that $\left|f_k(y) - b_k\right| < \frac{\epsilon}{3}$ and $\left|f_\ell(y) - b_\ell\right| < \frac{\epsilon}{3}$. Then we have

$$\left|b_k - b_\ell\right| \leq \left|b_k - f_k(y)\right| + \left|f_k(y) - f_\ell(y)\right| + \left|f_\ell(y) - b_\ell\right| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon\,.$$

By the Cauchy Criterion for sequences, (b_n) converges, as claimed.

Now, let $b = \lim_{n\to\infty} b_n$. We must show that $\lim_{y\to x} f(y) = b$. Let $\epsilon > 0$. Since $f_n \to f$ uniformly, we can choose m_1 so that $n \geq m_1 \Longrightarrow \left|f_n(y) - f(y)\right| < \frac{\epsilon}{3}$ for all $y \in A$, and since $b_n \to b$ we can choose m_2 so $n \geq m_2 \Longrightarrow |b_n - b| < \frac{\epsilon}{3}$. Let $m = \max\{m_1, m_2\}$. Let $n \geq m$. Since $\lim_{y\to x} f_n(y) = b_n$ we can choose $\delta > 0$ so that $0 < |y - x| < \delta \Longrightarrow \left|f_n(y) - b_n\right| < \frac{\epsilon}{3}$. Then when $0 < |y - x| < \delta$ we have

$$\left|f(y) - b\right| \leq \left|f(y) - f_n(y)\right| + \left|f_n(y) - b_n\right| + \left|b_n - b\right| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon\,.$$

Thus $\lim_{y\to x} f(y) = b$, as required.

In particular, if $x \in A$ and each f_n is continuous at x then we have

$$\lim_{y\to x} f(y) = \lim_{y\to x}\lim_{n\to\infty} f_n(y) = \lim_{n\to\infty}\lim_{y\to x} f_n(y) = \lim_{n\to\infty} f_n(x) = f(x)$$

so f is continuous at x.

3.10 Theorem: *(Uniform Convergence and Integration) Suppose that $f_n \to f$ uniformly on $[a,b]$. If each f_n is integrable on $[a,b]$ then so is f. In this case, if $g_n(x) = \int_a^x f_n(t)\,dt$ and $g(x) = \int_a^x f(t)\,dt$, then $g_n \to g$ uniformly on $[a,b]$. In particular, we have*

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx\,.$$

Proof: Suppose that each f_n is integrable on $[a,b]$. We claim that f is integrable on $[a,b]$. Let $\epsilon > 0$. Choose N so that $n \geq N \Longrightarrow \left|f_n(x) - f(x)\right| < \frac{\epsilon}{4(b-a)}$ for all $x \in [a,b]$. Fix $n \geq N$. Choose a partition X of $[a,b]$ so that $U(f_n, X) - L(f_n, X) < \frac{\epsilon}{2}$. Note that since $\left|f_n(x) - f(x)\right| < \frac{\epsilon}{4(b-a)}$ we have $M_i(f) < M_i(f_n) + \frac{\epsilon}{4(b-a)}$ and $m_i(f) > m_i(f_n) - \frac{\epsilon}{4(b-a)}$, and so

$$\begin{aligned} U(f,X) - L(f,X) &= \sum_{i=1}^n \big(M_i(f) - m_i(f)\big)\Delta_i x < \sum_{i=1}^n \left(M_i(f_n) - m_i(f_n) + \tfrac{\epsilon}{2(b-a)}\right)\Delta_i x \\ &= U(f_n,X) - L(f_n,X) + \tfrac{\epsilon}{2} < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,. \end{aligned}$$

Thus f is integrable on $[a,b]$.

Now define $g_n(x) = \int_a^x f_n(t)\,dt$ and $g(x) = \int_a^x f(t)\,dt$. We claim that $g_n \to g$ uniformly on $[a,b]$. Let $\epsilon > 0$. Choose N so that $n \geq N \Longrightarrow \left|f_n(t) - f(t)\right| < \frac{\epsilon}{2(b-a)}$ for all $t \in I$. Let $n \geq N$. Let $x \in [a,b]$. Then we have

$$\begin{aligned} \left|g_n(x) - g(x)\right| &= \left|\int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt\right| = \left|\int_a^x f_n(t) - f(t)\ dt\right| \\ &\leq \int_a^x \left|f_n(t) - f(t)\right| dt \leq \int_a^x \tfrac{\epsilon}{2(b-a)}\ dt = \tfrac{\epsilon}{2(b-a)}(x-a) \leq \tfrac{\epsilon}{2} < \epsilon\,. \end{aligned}$$

Thus $g_n \to g$ uniformly on $[a,b]$, as required.

In particular, we have $\lim_{n\to\infty} g_n(b) = g(b)$, that is

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx\,.$$

3.11 Theorem: *(Uniform Convergence and Differentiation) Let (f_n) be a sequence of functions on $[a,b]$. Suppose that each f_n is differentiable on $[a,b]$, $\big({f_n}'\big)$ converges uniformly on $[a,b]$, and $\big(f_n(c)\big)$ converges for some $c\in[a,b]$. Then (f_n) converges uniformly on $[a,b]$, $\lim\limits_{n\to\infty} f_n(x)$ is differentiable, and*

$$\frac{d}{dx}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{d}{dx} f_n(x)\,.$$

Proof: We claim that (f_n) converges uniformly on $[a,b]$. Let $\epsilon>0$. Choose N so that when $n,m\geq N$ we have $\big|{f_n}'(t)-{f_m}'(t)\big|<\frac{\epsilon}{2(b-a)}$ for all $t\in[a,b]$ and we have $\big|f_n(c)-f_m(c)\big|<\frac{\epsilon}{2}$. Let $n,m\geq N$. Let $x\in[a,b]$. By the Mean Value Theorem applied to the function $f_n(x)-f_m(x)$, we can choose t between c and x so that

$$\big(f_n(x)-f_m(x)-f_n(c)+f_m(c)\big)=\big({f_n}'(t)-{f_m}'(t)\big)(x-c)\,.$$

Then we have

$$\begin{aligned}\big|f_n(x)-f_m(x)\big| &\leq \big|f_n(x)-f_m(x)-f_n(c)+f_m(c)\big|+\big|f_n(c)-f_m(c)\big|\\ &=\big|{f_n}'(t)-{f_m}'(t)\big||x-c|+\big|f_n(c)-f_m(c)\big|\\ &<\tfrac{\epsilon}{2(b-a)}(b-a)+\tfrac{\epsilon}{2}=\epsilon\,.\end{aligned}$$

Thus (f_n) converges uniformly on $[a,b]$.

Let $f(x)=\lim\limits_{n\to\infty} f_n(x)$. We claim that f is differentiable with $f'(x)=\lim\limits_{n\to\infty}{f_n}'(x)$ for all $x\in[a,b]$. Fix $x\in[a,b]$. Note that

$$\begin{aligned} f'(x)=\lim_{n\to\infty}{f_n}'(x) &\iff \lim_{y\to x}\frac{f(y)-f(x)}{y-x}=\lim_{n\to\infty}\lim_{y\to x}\frac{f_n(y)-f_n(x)}{y-x}\\ &\iff \lim_{y\to x}\lim_{n\to\infty}\frac{f_n(y)-f_n(x)}{y-x}=\lim_{n\to\infty}\lim_{y\to x}\frac{f_n(y)-f_n(x)}{y-x}\end{aligned}$$

so it suffices to show that $\big(g_n\big)$ converges uniformly on $[a,b]\setminus\{x\}$, where

$$g_n(y)=\frac{f_n(y)-f_n(x)}{y-x}\,.$$

Let $\epsilon>0$. Choose N so that $n,m\geq N\implies\big|{f_n}'(t)-{f_m}'(t)\big|<\epsilon$ for all $t\in[a,b]$. Let $n,m\geq N$. Let $y\in[a,b]\setminus\{x\}$. By the Mean Value Theorem, we can choose t between x and y so that

$$\big(f_n(y)-f_m(y)-f_n(x)+f_m(x)\big)=\big({f_n}'(t)-{f_m}'(t)\big)(y-x)\,.$$

Then

$$\big|g_n(y)-g_m(y)\big|=\left|\frac{f_n(y)-f_m(y)-f_n(x)+f_m(x)}{y-x}\right|=\big|{f_n}'(t)-{f_m}'(t)\big|<\epsilon\,.$$

Thus $\big(g_n\big)$ converges uniformly on $[a,b]\setminus\{x\}$, as required.

Series of Functions

3.12 Definition: Let $(f_n)_{n\geq p}$ be a sequence of functions on $A \subseteq \mathbb{R}$. The **series of functions** $\sum_{n\geq p} f_n(x)$ is defined to be the sequence $\big(S_l(x)\big)$ where $S_l(x) = \sum_{n=p}^{l} f_n(x)$. The function $S_l(x)$ is called the l^{th} **partial sum** of the series. We say the series $\sum_{n\geq p} f_n(x)$ converges pointwise (or uniformly) on A when the sequence $\{S_l\}$ converges, pointwise (or uniformly) on A. In this case, the **sum** of the series of functions is defined to be the function

$$f(x) = \sum_{n=p}^{\infty} f_n(x) = \lim_{l\to\infty} S_l(x)\,.$$

3.13 Theorem: *(Cauchy Criterion for the Uniform Convergence of a Series of Functions) The series $\sum_{n\geq p} f_n(x)$ converges uniformly (to some function f) on A if and only if for every $\epsilon > 0$ there exists $N \geq p$ such that for all $x \in A$ and for all $k, \ell \geq p$ we have*

$$\ell > k \geq N \Longrightarrow \left|\sum_{n=k+1}^{\ell} f_n(x)\right| < \epsilon\,.$$

Proof: This follows immediately from the analogous theorem for sequences of functions.

3.14 Theorem: *(Uniform Convergence, Limits and Continuity) Suppose that $\sum_{n\geq p} f_n(x)$ converges uniformly on A. Let x be a limit point of A. If $\lim_{y\to x} f_n(y)$ exists for all $n \geq p$, then*

$$\lim_{y\to x} \sum_{n=p}^{\infty} f_n(y) = \sum_{n=p}^{\infty} \lim_{y\to x} f_n(y)\,.$$

In particular, if each $f_n(x)$ is continuous on A then so is $\sum_{n=p}^{\infty} f_n(x)$.

Proof: This follows immediately from the analogous theorem for sequences of functions.

3.15 Theorem: *(Uniform Convergence and Integration) Suppose that $\sum_{n\geq p} f_n(x)$ converges uniformly on $[a, b]$. If each $f_n(x)$ is integrable on $[a, b]$, then so is $\sum_{n=p}^{\infty} f_n(x)$. In this case, if we define $g_n(x) = \int_a^x f_n(t)\,dt$ and $g(x) = \int_a^x \sum_{n=p}^{\infty} f_n(t)\,dt$, then $\sum_{n\geq p} g_n(x)$ converges uniformly to $g(x)$ on A. In particular, we have*

$$\int_a^b \sum_{n=p}^{\infty} f_n(x)\,dx = \sum_{n=p}^{\infty} \int_a^b f_n(x)\,dx\,.$$

Proof: This follows immediately from the analogous theorem for sequences of functions.

3.16 Theorem: *(Uniform Convergence and Differentiation) Suppose that each* $f_n(x)$ *is differentiable on* $[a,b]$, $\sum_{n\geq p} {f_n}'(x)$ *converges uniformly on* $[a,b]$, *and* $\sum_{n\geq p} f_n(c)$ *converges for some* $c\in[a,b]$. *Then* $\sum_{n\geq p} f_n(x)$ *converges uniformly on* $[a,b]$ *and*

$$\frac{d}{dx}\sum_{n=p}^{\infty} f_n(x) = \sum_{n=p}^{\infty}\frac{d}{dx}f_n(x)\,.$$

Proof: This follows immediately from the analogous theorem for sequences of functions.

3.17 Theorem: *(The Weierstrass M-Test) Suppose that* f_n *is bounded with* $|f_n(x)|\leq M_n$ *for all* $n\geq p$ *and* $x\in A$, *and* $\sum_{n\geq p} M_n$ *converges. Then* $\sum_{n\geq p} f_n(x)$ *converges uniformly on* A.

Proof: Let $\epsilon>0$. Choose N so that $\ell>k\geq N \implies \sum\limits_{n=k+1}^{\ell} M_n<\epsilon$. Let $\ell>k\geq N$. Let $x\in A$. Then

$$\left|\sum_{n=k+1}^{\ell} f_n(x)\right| \leq \sum_{n=k+1}^{\ell}\big|f_n(x)\big| \leq \sum_{n=k+1}^{\ell} M_n<\epsilon\,.$$

3.18 Example: Find a sequence of functions $\big(f_n(x)\big)_{n\geq 0}$, each of which is differentiable on $\mathbb{R}$, such that $\sum_{n\geq 0} f_n(x)$ converges uniformly on $\mathbb{R}$, but the sum $f(x)=\sum\limits_{n=0}^{\infty} f_n(x)$ is nowhere differentiable.

Solution: Let $f_n(x)=\frac{1}{2^n}\sin^2(8^n x)$. Since $|f_n(x)|\leq\frac{1}{2^n}$ and $\sum\frac{1}{2^n}$ converges, $\sum_{n\geq 0} f_n(x)$ converges uniformly on $\mathbb{R}$. Let $f(x)=\sum\limits_{n=0}^{\infty} f_n(x)$. We claim that $f(x)$ is nowhere differentiable. Let $x\in\mathbb{R}$. For each n, let m, a_n and b_n be such that $a_n=\frac{m\pi}{2\cdot 8^n}$, $b_n=\frac{(m+1)\pi}{2\cdot 8^n}$ and $x\in[a_n,b_n)$. Note that one of $f_n(a_n)$ and $f_n(b_n)$ is equal to $\frac{1}{2^n}$ and the other is equal to 0 so we have $\big|f_n(b_n)-f_n(a_n)\big|=\frac{1}{2^n}$. Note also that for $k>n$ we have $f_k(a_n)=f_k(b_n)=0$. Also, for all k we have $f_k(x)=\frac{1}{2^k}\sin^2(8^k x)$, ${f_k}'(x)=4^k\sin(2\cdot 8^k x)$, and $\big|{f_k}'(x)\big|\leq 4^k$, so by the Mean Value Theorem,

$$\big|f_k(b_n)-f_k(a_n)\big|\leq 4^k|b_n-a_n|\,.$$

Finally, note that if $f'(x)$ did exist, then we would have $f'(x)=\lim\limits_{n\to\infty}\dfrac{f(b_n)-f(a_n)}{b_n-a_n}$, but

$$\begin{aligned}\left|\frac{f(b_n)-f(a_n)}{b_n-a_n}\right| &= \left|\sum_{k=0}^{\infty}\frac{f_k(b_n)-f_k(a_n)}{b_n-a_n}\right| = \left|\sum_{k=0}^{n}\frac{f_k(b_n)-f_k(a_n)}{b_n-a_n}\right| \\ &\geq \left|\frac{f_n(b_n)-f_n(a_n)}{b_n-a_n}\right| - \sum_{k=0}^{n-1}\left|\frac{f_k(b_n)-f_k(a_n)}{b_n-a_n}\right| \\ &\geq \frac{\frac{1}{2^n}}{\frac{\pi}{2\cdot 8^n}} - \sum_{k=0}^{n-1}4^k = \tfrac{2\cdot 4^n}{\pi}-\tfrac{4^n-1}{3} = \left(\tfrac{2}{\pi}-\tfrac{1}{3}\right)4^n+\tfrac{1}{3}\to\infty \text{ as } n\to\infty\end{aligned}$$

Power Series

3.19 Definition: A **power series centred at** a is a series of the form $\sum_{n\geq 0} a_n(x-a)^n$ for some real numbers a_n, where we use the convention that $(x-a)^0 = 1$.

3.20 Example: The geometric series $\sum_{n\geq 0} x^n$ is a power series centred at 0. It converges when $|x| < 1$ and for all such x the sum of the series is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\,.$$

3.21 Lemma: *(Abel's Formula) Let $\{a_n\}$ and $\{b_n\}$ be sequences. Then we have*

$$\sum_{n=m}^{l} a_n b_n + \sum_{p=m}^{l-1}\left(\sum_{n=m}^{p} a_n\right)(b_{p-1} - b_p) = \left(\sum_{n=m}^{l} a_n\right) b_l\,.$$

Proof: We have

$$\begin{aligned}
\sum_{p=m}^{l-1}\left(\sum_{k=m}^{p} a_n\right)(b_{p+1} - b_p) =\ & a_m(b_{m+1} - b_m) + (a_m + a_{m+1})(b_{m+2} - b_{m+1}) \\
& + (a_m + a_{m+1} + a_{m+2})(b_{m+3} - b_{m+2}) \\
& + \cdots + (a_m + a_{m+1} + a_{m+2} + \cdots + a_{l-1})(b_l - b_{l-1}) \\
=\ & -a_m b_m - a_{m+1} b_{m+1} - \cdots - a_{l-1} b_{l-1} \\
& + (a_m + a_{m+1} + \cdots a_{l-1}) b_l - a_l b_l + a_l b_l \\
=\ & \left(\sum_{n=m}^{l} a_n\right) b_l - \sum_{n=m}^{l} a_n b_n\,.
\end{aligned}$$

3.22 Definition: Let (a_n) be a sequence in $\mathbb{R}$. We define $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} s_n$ where $s_n = \sup\{a_k \,|\, k \geq n\}$ (with $\limsup_{n\to\infty} a_n = \infty$ when (a_n) is not bounded above).

3.23 Theorem: *(The Interval and Radius of Convergence) Let $\sum_{n\geq 0} a_n(x-a)^n$ be a power series and let $R = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} \in [0,\infty]$. Then the set of $x \in \mathbb{R}$ for which the power series converges is an interval I centred at a of radius R. Indeed*

(1) if $|x-a| > R$ then $\lim_{n\to\infty} a_n(x-a) \neq 0$ so $\sum_{n\geq 0} n_n(x-a)^n$ diverges,

(2) if $|x-a| < R$ then $\sum_{n\geq 0} a_n(x-a)^n$ converges absolutely,

(3) if $0 < r < R$ then $\sum_{n\geq 0} a_n(x-a)^n$ converges uniformly in $[a-r, a+r]$, and

(4) (Abel's Theorem) if $\sum_{n\geq 0} a_n(x-a)^n$ converges when $x = a+R$ then the convergence is uniform on $[a, a+R]$, and similarly if $\sum_{n\geq 0} a_n(x-a)^n$ converges when $x = a-R$ then the convergence is uniform on $[a-R, a]$.

Proof: To prove part (1), suppose that $|x-a|>R$. Then

$$\limsup_{n\to\infty}\sqrt[n]{|a_n(x-a)^n|}=|x-a|\limsup_{n\to\infty}\sqrt[n]{|a_n|}>R\cdot\tfrac{1}{R}=1\,,$$

and so $\lim\limits_{n\to\infty}a_n(x-a)^n\neq 0$ and $\sum a_n(x-a)^n$ diverges, by the Root Test.

To prove part (2), suppose that $|x-a|<R$. Then

$$\limsup_{n\to\infty}\sqrt[n]{|a_n(x-a)^n|}=|x-a|\limsup_{n\to\infty}\sqrt[n]{|a_n|}<R\cdot\tfrac{1}{R}=1\,,$$

and so $\sum\big|a_n(x-a)^n\big|$ converges, by the Root Test.

To prove part (3), fix $0<r<R$. By part (2), $\sum\big|a_n(x-a)^n\big|$ converges when $x=a+r$, that is $\sum|a_nr^n|$ converges. Let $x\in[a-r,a+r]$. Then $|a_n(x-a)^n|\leq|a_nr^n|$ and $\sum|a_nr^n|$ converges, and so $\sum|a_n(x-a)^n|$ converges uniformly by the Weierstrass M-Test.

Now let us prove part (4). Suppose that $\sum a_n(x-a)^n$ converges when $x=a+R$, that is $\sum a_nR^n$ converges. Let $\epsilon>0$. Choose N so that $l>m>N\Longrightarrow\left|\sum\limits_{n=m}^{l}a_nR^n\right|<\epsilon$.

Then by Abel's Formula and using telescoping we have

$$\begin{aligned}\left|\sum_{n=m}^{l}a_n(x-a)^n\right|&=\left|\sum_{n=m}^{l}a_nR^n\left(\tfrac{x-a}{R}\right)^n\right|\\&=\left|\left(\sum_{n=m}^{l}a_nR^n\right)\left(\tfrac{x-a}{R}\right)^l-\sum_{p=m}^{l-1}\left(\sum_{n=m}^{p}a_nR^n\right)\left(\left(\tfrac{x-a}{R}\right)^{p+1}-\left(\tfrac{x-a}{R}\right)^p\right)\right|\\&\leq\left|\sum_{n=m}^{l}a_nR^n\right|\left(\tfrac{x-a}{R}\right)^l+\sum_{p=m}^{l-1}\left|\sum_{n=m}^{p}a_nR^n\right|\left(\left(\tfrac{x-a}{R}\right)^p-\left(\tfrac{x-a}{R}\right)^{p+1}\right)\\&<\epsilon\left(\tfrac{x-a}{R}\right)^l+\epsilon\left(\left(\tfrac{x-a}{R}\right)^m-\left(\tfrac{x-a}{R}\right)^l\right)=\epsilon\left(\tfrac{x-a}{R}\right)^m<\epsilon\,.\end{aligned}$$

3.24 Definition: The number R in the above theorem is called the **radius of convergence** of the power series, and the interval I is called the **interval of convergence** of the power series.

3.25 Example: Find the interval of convergence of the power series $\displaystyle\sum_{n\geq 1}\frac{(3-2x)^n}{\sqrt{n}}$.

Solution: First note that this is in fact a power series, since $\dfrac{(3-2x)^n}{\sqrt{n}}=\frac{(-2)^n}{\sqrt{n}}\left(x-\frac{3}{2}\right)^n$, and so $\displaystyle\sum_{n\geq 1}\frac{(3-2x)^n}{\sqrt{n}}=\sum_{n\geq 0}c_n(x-a)^n$, where $c_0=0$, $c_n=\frac{(-2)^n}{\sqrt{n}}$ for $n\geq 1$ and $a=\frac{3}{2}$.

Now, let $a_n=\dfrac{(3-2x)^n}{\sqrt{n}}$. Then $\left|\dfrac{a_{n+1}}{a_n}\right|=\left|\dfrac{(3-2x)^{n+1}}{\sqrt{n+1}}\,\dfrac{\sqrt{n}}{(3-2x)^n}\right|=\sqrt{\frac{n}{n+1}}\,|3-2x|$, so $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=|3-2x|$. By the Ratio Test, $\sum a_n$ converges when $|3-2x|<1$ and diverges when $|3-2x|>1$. Equivalently, it converges when $x\in(1,2)$ and diverges when $x\notin[1,2]$. When $x=1$ so $(3-2x)=1$, we have $\sum a_n=\sum\frac{1}{\sqrt{n}}$, which diverges (its a p-series), and when $x=2$ so $(3-2x)=-1$, we have $\sum a_n=\sum\frac{(-1)^n}{\sqrt{n}}$ which converges by the Alternating Series Test. Thus the interval of convergence is $I=(1,2\,]$.

Operations on Power Series

3.26 Theorem: *(Continuity of Power Series) Suppose that the power series* $\sum a_n(x-a)^n$ *converges in an interval* I*. Then the sum* $f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n$ *is continuous in* I*.*

Proof: This follows from uniform convergence of $\sum a_n(x-a)^n$ in closed subintervals of I.

3.27 Theorem: *(Addition and Subtraction of Power Series) Suppose that the power series* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in the interval* I*. Then* $\sum(a_n+b_n)(x-a)^n$ *and* $\sum(a_n-b_n)(x-a)^n$ *both converge in* I*, and for all* $x\in I$ *we have*

$$\left(\sum_{n=0}^{\infty}a_n(x-a)^n\right)\pm\left(\sum_{n=0}^{\infty}b_n(x-a)^n\right)=\sum_{n=0}^{\infty}(a_n\pm b_n)(x-a)^n\,.$$

Proof: This follows from Linearity.

3.28 Theorem: *(Multiplication of Power Series) Suppose the power series* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in an open interval* I *with* $a\in I$*. Let* $c_n=\sum_{k=0}^{n}a_kb_{n-k}$*. Then* $\sum c_n(x-a)^n$ *converges in* I *and for all* $x\in I$ *we have*

$$\sum_{n=0}^{\infty}c_n(x-a)^n=\left(\sum_{n=0}^{\infty}a_n(x-a)^n\right)\left(\sum_{n=0}^{\infty}b_n(x-a)^n\right)\,.$$

Proof: This follows from the Multiplication of Series Theorem, since the power series converge absolutely in I.

3.29 Theorem: *(Division of Power Series) Suppose that* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in an open interval* I *with* $a\in I$*, and that* $b_0\neq 0$*. Define* c_n *by*

$$c_0=\tfrac{a_0}{b_0}\ \text{, and for } n>0,\ c_n=\tfrac{a_n}{b_0}-\tfrac{b_nc_0}{b_0}-\tfrac{b_{n-1}c_1}{b_0}-\cdots-\tfrac{b_1c_{n-1}}{b_0}\,.$$

Then there is an open interval J *with* $a\in J$ *such that* $\sum c_n(x-a)^n$ *converges in* J *and for all* $x\in J$*,*

$$\sum_{n=0}^{\infty}c_n(x-a)^n=\frac{\displaystyle\sum_{n=0}^{\infty}a_n(x-a)^n}{\displaystyle\sum_{n=0}^{\infty}b_n(x-a)^n}\,.$$

Proof: Choose $r>0$ so that $a+r\in I$. Note that $\sum|a_nr^n|$ and $\sum|b_nr^n|$ both converges. Since $|a_nr^n|\to 0$ and $|b_nr^n|\to 0$ and $b_0\neq 0$, we can choose M so that $M\geq\left|\frac{a_nr^n}{b_0}\right|$ and $M\geq\left|\frac{b_nr^n}{b_0}\right|$ for all n. Note that $|c_0|=\left|\frac{a_0}{b_0}\right|\leq M$ and since $c_1=\frac{a_1}{b_0}+\frac{b_1c_0}{b_0}$ we have

$$|c_1r|\leq\left|\tfrac{a_1r}{b_0}\right|+\left|\tfrac{b_1r}{b_0}\right||c_0|\leq M+M^2=M(1+M)\,.$$

Suppose, inductively, that $|c_kr^k|\leq M(1+M)^k$ for all $k<n$. Then since

$$a_n=b_nc_0+b_{n-1}c_1+\cdots+b_1c_{n-1}+b_0c_n\,,$$

we have

$$\begin{aligned}|c_n r^n| &\le \left|\tfrac{a_n r^n}{b_0}\right| + \left|\tfrac{b_n r^n}{b_0}\right| |c_0| + \left|\tfrac{b_{n-1} r^{n-1}}{b_0}\right| |c_1 r| + \cdots + \left|\tfrac{b_1 r}{b_0}\right| |c_{n-1} r^{n-1}| \\ &\le M + M^2 + M^2(1+M) + M^2(1+M)^2 + M^2(1+M)^3 + \cdots + M^2(1+M)^{n-1} \\ &= M + M^2\left(\frac{(1+M)^n - 1}{M}\right) = M(1+M)^n \,.\end{aligned}$$

Bu induction, we have $|c_n r^n| \le M(1+M)^n$ for all $n \ge 0$. Let $J_1 = \left(a - \frac{r}{1+M}, a + \frac{r}{1+M}\right)$. Let $x \in J_1$ so $|x-a| < \frac{r}{1+M}$. Then for all n we have

$$|c_n(x-a)^n| = |c_n r^n| \cdot \frac{1}{(1+M)^n} \cdot \left|\frac{x-a}{r/(1+M)}\right|^n \le M \left|\frac{x-a}{r/(1+M)}\right|^n$$

and so $\sum |c_n(x-a)^n|$ converges by the Comparison Test.

Note that from the definition of c_n we have $a_n = \sum\limits_{k=0}^{n} c_k b_{n-k}$, and so by multiplying power series, we have

$$\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-a)^n\right) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for all $x \in I \cap J_1$. Finally note that $f(x) = \sum\limits_{n=0}^{\infty} b_n(x-a)^n$ is continuous in I and we have $f(0) = b_0 \ne 0$, and so there is an interval $J \subset I \cap J_1$ with $a \in J$ such that $f(x) \ne 0$ in J.

3.30 Theorem: *(Composition of Power Series) Let $f(x) = \sum\limits_{n=0}^{\infty} a_n(x-a)^n$ in an open interval I with $a \in I$, and let $g(y) = \sum\limits_{m=0}^{\infty} b_m(y-b)^m$ in an open interval J with $b \in J$ and with $a_0 \in J$. Let K be an open interval with $a \in K$ such that $f(K) \subset J$. For each $m \ge 0$, let $c_{n,m}$ be the coefficients, found by multiplying power series, such that $\sum\limits_{n=0}^{\infty} c_{n,m}(x-a)^n = b_n\left(\sum\limits_{n=0}^{\infty} a_n(x-a)^n - b\right)^m$. Then $\sum\limits_{m\ge 0} c_{n,m}$ converges for all $m \ge 0$, and for all $x \in K$, $\sum\limits_{n\ge 0}\Big(\sum\limits_{m=0}^{\infty} c_{n,m}\Big)(x-a)^n$ converges and*

$$\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty} c_{n,m}\Big)(x-a)^n = g\big(f(x)\big)\,.$$

Proof: This follows from Fubini's Theorem for Series since

$$g\big(f(x)\big) = \sum_{m=0}^{\infty} b_m\big(f(x)-b\big)^m = \sum_{m=0}^{\infty} b_m\left(\sum_{n=0}^{\infty} a_n(x-a)^n - b\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} c_{n,m}(x-a)^n\right).$$

3.31 Theorem: *(Integration of Power Series) Suppose that $\sum a_n(x-a)^n$ converges in the interval I. Then for all $x\in I$, the sum $f(x)=\sum\limits_{n=0}^{\infty}a_n(x-a)^n$ is integrable on $[a,x]$ (or $[x,a]$) and*

$$\int_a^x\sum_{n=0}^{\infty}a_n(t-a)^n\,dt=\sum_{n=0}^{\infty}\int_a^x a_n(t-a)^n\,dt=\sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-a)^{n+1}\,.$$

Proof: This follows from uniform convergence.

3.32 Theorem: *(Differentiation of Power Series) Suppose that $\sum a_n(x-a)^n$ converges in the open interval I. Then the sum $f(x)=\sum\limits_{n=0}^{\infty}a_n(x-a)^n$ is differentiable in I and*

$$f'(x)=\sum_{n=1}^{\infty}n\,a_n(x-a)^{n-1}\,.$$

Proof: We claim that the radius of convergence of $\sum a_n(x-a)^n$ is equal to the radius of convergence of $\sum na_n(x-a)^{n-1}$. Let R be the radius of convergence of $\sum a_n(x-a)^n$ and let S be the radius of convergence of $\sum na_n(x-a)^{n-1}$. Fix $x\in(a-R,a+R)$ so $|x-a|<R$ and $\sum\left|a_n(x-a)^n\right|$ converges. Choose r,s with $|x-a|<r<s<R$. Since $\lim\limits_{n\to\infty}\dfrac{(r/s)^n}{(1/n)}=0$, we can choose N so that $n\geq N\implies\left(\frac{r}{s}\right)^n<\frac{1}{n}$. Then for $n\geq N$ we have

$$\left|na_n(x-a)^n\right|=\left|n\left(\tfrac{r}{s}\right)^n\left(\tfrac{x-a}{r}\right)^n a_ns^n\right|\leq 1\cdot 1\cdot|a_ns^n|\,.$$

Since $\sum|a_ns^n|$ converges, $\sum\left|na_n(x-a)^n\right|$ converges by the Comparison Test, and so $\sum\left|na_n(x-a)^{n-1}\right|$ converges by Linearity. Thus $R\leq S$.

Now fix $x\in(a-S,a+s)$ so that $|x-a|<S$ and $\sum\left|na_n(x-a)^{n-1}\right|$ converges. Then $\sum\left|na_n(x-a)^n\right|$ converges by Linearity, and $\left|a_n(x-a)^n\right|\leq\left|na_n(x-a)^n\right|$ so $\sum\left|a_n(x-a)^n\right|$ converges by Comparison. Thus $S\leq R$ and so $R=S$ as claimed.

The theorem now follows from the uniform convergence of $\sum na_n(x-a)^{n-1}$.

3.33 Example: We have $\frac{1}{1+x}=\sum\limits_{n=0}^{\infty}(-1)^nx^n$ for $|x|<1$. By Integration of Power Series, $\ln x=\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}=\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n}x^n$ for $|x|<1$. In particular, we can take $x=\frac{1}{2}$ to get $\ln\frac{3}{2}=\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n\cdot 2^n}$ and we can take $x=-\frac{1}{2}$ to get $\ln\frac{1}{2}=\sum\limits_{n=1}^{\infty}\frac{-1}{n\cdot 2^n}$, that is $\ln 2=\sum\limits_{n=1}^{\infty}\frac{1}{n\cdot 2^n}$.

Let us also argue that we can also take $x=1$. Note that the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$ diverges when $x=-1$ (by the Integral Test) and converges when $x=1$ (by the Alternating Series Test), so the interval of convergence is $(-1,1]$. Thus the sum $f(x)=\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$ is defined for $-1<x\leq 1$. We know already that $f(x)=\ln(1+x)$ for $-1<x<1$. By Abel's Theorem, the series converges uniformly on $[0,1]$, so by the Continuity of Power Series Theorem, the sum $f(x)=\sum\limits_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}x^n$ is continuous on $[0,1]$ and in particular $f(x)$ is continuous at $x=1$. Since $f(x)=\ln(1+x)$ for $|x|<1$ and and since both $f(x)$ and $\ln(1+x)$ are continuous at 1 it follows that $f(1)=\ln 2$. Thus we have $\ln 2=\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$.

Taylor Series

3.34 Theorem: *Suppose that* $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ *in an open interval* I *centred at* a. *Then* f *is infinitely differentiable at* a *and for all* $n \geq 0$ *we have*

$$a_n = \frac{f^{(n)}(a)}{n!}\,,$$

where $f^{(n)}(a)$ *denotes the* n^{th} *derivative of* f *at* a.

Proof: By repeated application of the Differentiation of Power Series Theorem, for all $x \in I$, we have $f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$, $f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-a)^{n-2}$ and $f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n (x-a)^{n-3}$, and in general

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (x-a)^{n-k}$$

and so $f(a) = a_0$, $f'(a) = a_1$, $f''(a) = 2\cdot 1\, a_2$ and $f'''(a) = 3\cdot 2\cdot 1\, a_3$, and in general

$$f^{(n)}(a) = n!\; a_n$$

3.35 Definition: Given a function $f(x)$ whose derivatives of all order exist at $x = a$, we define the **Taylor series** of $f(x)$ centered at a to be the power series

$$T(x) = \sum_{n\geq 0} a_n (x-a)^n \quad \text{where } a_n = \frac{f^{(n)}(a)}{n!}$$

and we define the l^{th} **Taylor Polynomial** of $f(x)$ centered at a to be the l^{th} partial sum

$$T_l(x) = \sum_{n=0}^{l} a_n (x-a)^n \quad \text{where } a_n = \frac{f^{(n)}(a)}{n!}$$

3.36 Example: Find the Taylor series centered at 0 for $f(x) = e^x$.

Solution: We have $f^{(n)}(x) = e^x$ for all n, so $f^{(n)}(0) = 1$ and $a_n = \frac{1}{n!}$ for all $n \geq 0$. Thus the Taylor series is

$$T(x) = \sum_{n=0}^{\infty} \tfrac{1}{n!}\, x^n = 1 + x + \tfrac{1}{2!}x^2 = \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \cdots\,.$$

3.37 Example: Find the Taylor series centered at 0 for $f(x) = \sin x$.

Solution: We have $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f''''(x) = \sin x$ and so on, so that in general $f^{(2n)}(x) = (-1)^n \sin x$ and $f^{(2n+1)}(x) = (-1)^n \cos x$. It follows that $f^{(2n)}(0) = 0$ and $f^{(2n+1)}(0) = (-1)^n$, so we have $a_{2n} = 0$ and $a_{2n+1} = \frac{(-1)^n}{(2n+1)!}$. Thus

$$T(x) = \sum_{n=0}^{\infty} \tfrac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots\,.$$

3.38 Example: Find the Taylor series centered at 0 for $f(x) = (1+x)^p$ where $p \in \mathbb{R}$.

Solution: $f'(x) = p(1+x)^{p-1}$, $f''(x) = p(p-1)(1+x)^{p-2}$, $f'''(x) = p(p-1)(p-2)(1+x)^{p-3}$, and in general

$$f^{(n)}(x) = p(p-1)(p-2)\cdots(p-n+1)(1+x)^{p-n}\,,$$

so $f(0) = 1$, $f'(0) = p$, $f''(0) = p(p-1)$, and in general $f^{(n)}(0) = p(p-1)(p-2)\cdots(p-n+1)$, and so we have $a_n = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$. Thus the Taylor series is

$$T(x) = \sum_{n=0}^{\infty} \binom{p}{n} x^n = 1 + px + \tfrac{p(p-1)}{2!}x^2 + \tfrac{p(p-1)(p-2)}{3!}x^3 + \tfrac{p(p-1)(p-2)(p-3)}{4!}x^4 + \cdots$$

where we use the notation

$$\binom{p}{0} = 1\ \text{, and for } n \geq 1,\ \binom{p}{n} = \tfrac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$$

3.39 Theorem: *(Taylor) Let $f(x)$ be infinitely differentiable in an open interval I with $a \in I$. Let $T_l(x)$ be the l^{th} Taylor polynomial for $f(x)$ centered at a. Then for all $x \in I$ there exists a number c between a and x such that*

$$f(x) - T_l(x) = \frac{f^{(l+1)}(c)}{(l+1)!}(x-a)^{l+1}\,.$$

Proof: When $x = a$ both sides of the above equation are 0. Suppose that $x > a$ (the case that $x < a$ is similar). Since $f^{(l+1)}$ is differentiable and hence continuous, by the Extreme Value Theorem it attains its maximum and minimum values, say M and m. Since $m \leq f^{(l+1)}(t) \leq M$ for all $t \in I$, we have

$$\int_a^{t_1} m\,dt \leq \int_a^{t_1} f^{(l+1)}(t)\,dt \leq \int_a^{t_1} M\,dt$$

that is

$$m(t_1 - a) \leq f^{(l)}(t_1) - f^{(l)}(a) \leq M(t_1 - a)$$

for all $t_1 > a$ in I. Integrating each term with respect to t_1 from a to t_2, we get

$$\tfrac{1}{2}m(t_2-a)^2 \leq f^{(l-1)}(t_2) - f^{(l)}(a)(t_2-a) \leq \tfrac{1}{2}M(t_t-a)^2$$

for all $t_2 > a$ in I. Integrating with respect to t_2 from a to t_3 gives

$$\tfrac{1}{3!}m(t_3-a)^3 \leq f^{(l-2)}(t_3) - f^{(l-2)}(a) - \tfrac{1}{2}f^{(l)}(a)(t_3-a)^3 \leq \tfrac{1}{3!}M(t_3-a)^3$$

for all $t_3 > a$ in I. Repeating this procedure eventually gives

$$\tfrac{1}{(l+1)!}m(t_{l+1}-a)^{l+1} \leq f(t_{l+1}) - T_l(t_{l+1}) \leq \tfrac{1}{(l+1)!}M(t_{l+1}-a)^{l+1}$$

for all $t_{l+1} > a$ in I. In particular $\frac{1}{(l+1)!}m(x-a)^{l+1} \leq f(x) - T_l(x) \leq \frac{1}{(l+1)!}M(x-a)^{l+1}$, so

$$m \leq \big(f(x) - T_l(x)\big)\tfrac{(l+1)!}{(x-a)^{l+1}} \leq M\,.$$

By the Intermediate Value Theorem, there is a number $c \in [a, x]$ such that

$$f^{(l+1)}(c) = \big(f(x) - T_l(x)\big)\frac{(l+1)!}{(x-a)^{l+1}}$$

.

3.40 Theorem: *The functions e^x, $\sin x$ and $(1+x)^p$ are all exactly equal to the sum of their Taylor series centered at 0 in the interval of convergence.*

Proof: First let $f(x)=e^x$ and let $x\in\mathbb{R}$. By Taylor's Theorem, $f(x)-T_l(x)=\dfrac{e^c x^{l+1}}{(l+1)!}$ for some c between 0 and x, and so

$$\big|f(x)-T_l(x)\big|\le\frac{e^{|x|}|x|^{l+1}}{(l+1)!}\,.$$

Since $\sum\dfrac{e^{|x|}|x|^{l+1}}{(l+1)!}$ converges by the Ratio Test, we have $\lim\limits_{l\to\infty}\dfrac{e^{|x|}|x|^{l+1}}{(l+1)!}=0$ by the Divergence Test, so $\lim\limits_{l\to\infty}\big(f(x)-T_l(x)\big)=0$, and so $f(x)=\lim\limits_{l\to\infty}T_l(x)=T(x)$.

Now let $f(x)=\sin x$ and let $x\in\mathbb{R}$. By Taylor's Theorem, $f(x)-T(x)=\dfrac{f^{(l+1)}(c)\,x^{l+1}}{(l+1)!}$ for some c between 0 and x. Since $f^{(l+1)}(x)$ is one of the functions $\pm\sin x$ or $\pm\cos x$, we have $\big|f^{(l+1)}(c)\big|\le 1$ for all c and so

$$\big|f(x)-T(x)\big|\le\frac{|x|^{l+1}}{(l+1)!}\,.$$

Since $\sum\dfrac{|x|^{l+1}}{(l+1)!}$ converges by the Ratio Test, $\lim\limits_{l\to\infty}\dfrac{|x|^{l+1}}{(l+1)!}=0$ by the Divergence Test, and so we have and $f(x)=T(x)$ as above.

Finally, let $f(x)=(1+x)^p$. The Taylor series centered at 0 is

$$T(x)=1+px+\tfrac{p(p-1)}{2!}x^2+\tfrac{p(p-1)(p-2)}{3!}x^3+\tfrac{p(p-1)(p-2)(p-3)}{4!}x^4+\cdots$$

and it converges for $|x|<1$. Differentiating the power series gives

$$T'(x)=p+\tfrac{p(p-1)}{1!}x+\tfrac{p(p-1)(p-2)}{2!}x^2+\tfrac{p(p-1)(p-2)(p-3)}{3!}x^3+\cdots$$

and so

$$\begin{aligned}(1+x)T'(x)&=p+\left(p+\tfrac{p(p-1)}{1!}\right)x+\left(\tfrac{p(p-1)}{1!}+\tfrac{p(p-1)(p-2)}{2!}\right)x^2\\&\quad+\left(\tfrac{p(p-1)(p-2)}{2!}-\tfrac{p(p-1)(p-2)(p-3)}{3!}\right)x^3+\cdots\\&=p+\tfrac{p\cdot p}{1!}x+\tfrac{p\cdot p(p-1)}{2!}x^2+\tfrac{p\cdot p(p-1)(p-2)}{3!}x^3+\cdots\\&=p\,T(x)\,.\end{aligned}$$

Thus we have $(1+x)T'(x)=pT(x)$ with $T(0)=1$. This DE is linear since we can write it as $T'(x)-\frac{p}{1+x}T(x)=0$. An integrating factor is $\lambda=e^{\int-\frac{p}{1+x}\,dx}=e^{-p\ln(1+x)}=(1+x)^{-p}$ and the solution is $T(x)=(1+x)^{-p}\displaystyle\int 0\,dx=b(1+x)^p$ for some constant b. Since $T(0)=1$ we have $b=1$ and so $T(x)=(1+x)^p=f(x)$.