# AMATH/PMATH 331 Solutions to Assignment 5.5

**1:** (a) Define $f:\mathbb{R}^2\to\mathbb{R}^2$ by $f(x,y)=\left(x,\frac{y}{2}\right)$. Show that $d\big(f(u),f(v)\big)\le d(u,v)$ for all $u,v\in\mathbb{R}^2$, determine whether $f$ is a contraction map, and determine whether $f$ has a unique fixed point in $\mathbb{R}^2$.

Solution: For $u=(a,b)$ and $v=(c,d)$ we have

$$d\big(f(u),f(v)\big)=\left|\left(a-c,\tfrac{1}{2}(b-d)\right)\right|=\sqrt{(a-b)^2+\tfrac{1}{4}(c-d)^2}\le\sqrt{(a-c)^2+(b-d)^2}=\big|(a-c,b-d)\big|=d(u,v).$$

Note that $f$ does not have a unique fixed point because $f(x,0)=(x,0)$ for all $x\in\mathbb{R}$. Since $\mathbb{R}^2$ is complete and $f$ does not have a unique fixed point, it cannot be a contraction map by the Banach Fixed Point Theorem.

(b) The polynomial $p(x)=x^3-3x+1$ has a unique root in $\left[0,\frac{1}{2}\right]$. Approximate this root using the Banach Fixed Point Theorem as follows: Let $f(x)=\frac{1}{3}(x^3+1)$. Show that $f:\left[0,\frac{1}{2}\right]\to\left[0,\frac{1}{2}\right]$ is a contraction map whose unique fixed point is the desired root of $p$. Approximate the root by using a calculator to find $x_5$ where $x_0=0$ and $x_{n+1}=f(x_n)$.

Solution: We have $f'(x)=x^2$. Since $f'(x)>0$ for $x>0$, it follows that $f$ is (strictly) increasing for $x\ge 0$ (see Corollary 1.56), and since $f(0)=\frac{1}{3}$ and $f\left(\frac{1}{2}\right)=\frac{3}{8}$ we have $f:\left[0,\frac{1}{2}\right]\to\left[\frac{1}{3},\frac{3}{8}\right]\subseteq\left[0,\frac{1}{2}\right]$ (see Problem 3(a) on Assignment 1). Let $x,y\in\left[0,\frac{1}{2}\right]$. By the Mean Value Theorem we can choose $t$ between $x$ and $y$ so that $f(x)-f(y)=f'(t)(x-y)$. Since $t$ is between $x$ and $y$, we have $0\le t\le\frac{1}{2}$ hence $0\le t^2\le\frac{1}{4}$, that is $0\le f'(t)\le\frac{1}{4}$. Thus $\big|f(x)-f(y)\big|=\big|f'(t)\big|\,|x-y|\le\frac{1}{4}\,|x-y|$ so that $f$ is a contraction map with contraction constant $c=\frac{1}{4}$. Using a calculator, we find that $x_0=0$, $x_1\cong 0.333333$, $x_2\cong 0.345679$, $x_3\cong 0.347102$, $x_4\cong 0.347273$ and $x_5\cong 0.347294$.

We remark that Newton's Method for finding this root (which many students will have seen) amounts to finding the fixed point of the contraction map $g(x)=x-\frac{p(x)}{p'(x)}=\frac{2x^3-1}{3(x^2-1)}$, and the resulting sequence converges faster than the sequence we found above (because, letting $a$ be the root that we are approximating, when we repeatedly apply $f$ on smaller intervals the contraction constant approaches $f'(a)=a^2\cong 0.12$ but $p$ is a factor of $g'$ so when we repeatedly apply $g$ the contraction constant approaches $g'(a)=0$).

We also remark that the exact value of the root that we are approximating is $a=2\sin\frac{\pi}{9}=2\sin(10^\circ)$.

**2:** (a) Define $F:\mathbb{R}^2\to\mathbb{R}$ by $F(x,y)=3\,y^{2/3}$. Determine whether $F$ satisfies the hypothesis of Picard's Theorem, whether $F$ satisfies the hypothesis of Peano's Theorem, and whether there exists $\delta>0$ such that the differential equation $\frac{dy}{dx}=F(x,y)$ has a unique solution $y=f(x)$ with $f(0)=0$, defined for all $x\in(-\delta,\delta)$.

Solution: The function $F$ does satisfy the hypothesis of Peano's Theorem, because it is continuous. There does not exist $\delta>0$ such that the differential equation $\frac{dy}{dx}=F(x,y)=3\,y^{2/3}$ has a unique solution $y=f(x)$ with $f(0)=0$, defined for all $x\in(-\delta,\delta)$ because, given $\delta>0$, for all $c\in[0,\delta)$ the function $f:(-\delta,\delta)\to\mathbb{R}$ given by $f(x)=0$ for $x\le c$ and $f(x)=(x-c)^3$ for $x\ge c$ is a solution. Since the differential equation does not have a unique solution, it follows that $F$ cannot satisfy the hypothesis of Picard's Theorem.

(b) Let $a_1,a_2,\cdots,a_n\in\mathbb{R}^n$ with say $a_k=(a_{k,1},a_{k,2},\cdots,a_{k,n})$, and let $A\in M_n(\mathbb{R})$ be that matrix with entries $a_{k,\ell}$. By applying the Banach Fixed Point Theorem to the map $F:(\mathbb{R}^n,d_\infty)\to(\mathbb{R}^n,d_\infty)$ given by $F(x)=Ax+b$, where $b\in\mathbb{R}^n$, show that if $\|a_k\|_1<1$ for all indices $k$ then the matrix $I-A$ is invertible.

Solution: Let $b\in\mathbb{R}^n$. Suppose that $\|a_k\|_1<1$ for all indices $k$. Choose $c\in[0,1)$ so that $\|a_k\|_\infty\le c$ for all $k$. For all $x,y\in\mathbb{R}^n$ we have

$$\begin{aligned}\big\|F(x)-F(y)\big\|_\infty&=\big\|A(x-y)\big\|_\infty=\max_{1\le k\le n}\sum_{i=1}^n a_{k,i}(x-y)_i\le\max_{1\le k\le n}\sum_{i=1}^n a_{k,i}\|x-y\|_\infty\\&\le\max_{1\le k\le n}\sum_{i=1}^n|a_{k,i}|\,\|x-y\|_\infty=\max_{1\le k\le n}\|a_k\|_1\cdot\|x-y\|_\infty\le c\,\|x-y\|_\infty\end{aligned}$$

so that $F:(\mathbb{R}^n,d_\infty)\to(\mathbb{R}^n,d_\infty)$ is a contraction map. Since $(\mathbb{R}^n,d_\infty)$ is complete and $F$ is a contraction map, it follows from the Banach Fixed Point Theorem that there is a unique element $x\in\mathbb{R}^n$ such that $x=F(x)=Ax+b$, that is a unique $x\in\mathbb{R}^n$ such that $(I-A)x=b$. Since the equation $(I-A)x=b$ has a solution for every $b\in\mathbb{R}^n$, it follows that the $n\times n$ matrix $I-A$ has rank $n$, so it is invertible.

**3:** (a) Find $D(A,B)$ where $D$ is the Hausdorff metric and $A$ is the line segment in $\mathbb{R}^2$ from $(0,0)$ to $(3,4)$ and $B$ is the line segment in $\mathbb{R}^2$ from $(0,1)$ to $(4,3)$.

Solution: For $a\in A$, the distance $d(a,B)$ is maximized when $a$ is an endpoint of $A$. It is clear that $d\big((0,0),B\big)=d\big((0,0),(0,1)\big)=1$. The line through $(3,4)$ perpendicular to $B$ is the line $y=10-2x$, and it meets the line containing $B$, which is given by given by $y=1+\frac{1}{2}x$, at the point $\left(\frac{18}{5},\frac{14}{5}\right)$ (which is in $B$), so we have $d\big((3,4),B\big)=d\big((3,4),\left(\frac{18}{5},\frac{14}{5}\right)\big)=\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{6}{5}\right)^2}=\frac{3}{\sqrt5}$. We have $\frac{3}{\sqrt5}>1$ (since $3>\sqrt5$) so $\max\limits_{a\in A}d(a,B)=\frac{3}{\sqrt5}$.

The line through $(0,1)$ perpendicular to $A$ is given by $y=1-\frac{3}{4}x$, and it meets the line containing $A$, which is given by $y=\frac{4}{3}x$, at the point $\left(\frac{12}{25},\frac{16}{25}\right)$ (which in in $A$), so we have $d\big((0,1),A\big)=d\big((0,1),\left(\frac{12}{25},\frac{16}{25}\right)=\sqrt{\left(\frac{12}{25}\right)^2+\left(\frac{9}{25}\right)^2}=\frac{3}{25}\sqrt{4^2+3^2}=\frac{3}{5}$. The line through $(3,4)$ perpendicular to $A$ is given by $y=6-\frac{3}{4}x$, and it meets the line through $A$ at $\left(\frac{72}{25},\frac{96}{25}\right)$ (which is in $A$), so we have $d\big((4,3),A\big)=d\big((4,3),\left(\frac{76}{25}\right),\left(\frac{96}{25}\right)\big)=\sqrt{\left(\frac{28}{25}\right)^2+\left(\frac{21}{25}\right)^2}=\frac{7}{25}\sqrt{4^2+3^3}=\frac{7}{5}$. Since $\frac{7}{5}>\frac{3}{5}$ we have $\max\limits_{b\in B}d(b,A)=\frac{7}{5}$. Note that $\frac{7}{5}>\frac{3}{\sqrt5}$ (because $\frac{7}{5}>\frac{3}{\sqrt5}\iff 7>3\sqrt5\iff 49>45$), so $D(A,B)=\max\left\{\max\limits_{a\in A}d(a,B),\max\limits_{b\in B}d(b,A)\right\}=\max\left\{\frac{3}{\sqrt5},\frac{7}{5}\right\}=\frac{7}{5}$.

(b) Let $K$ be the set of all nonempty compact sets in $\mathbb{R}^2$. Show that for every closed set $C\subseteq\mathbb{R}^2$ (in the standard metric), the set $S=\left\{A\in K\,\middle|\,A\subseteq C\right\}$ is closed in $K$, using the Hausdorff metric.

Solution: Let $C\subseteq\mathbb{R}^2$ be closed (in the standard metric) and let $S=\left\{A\in K\,\middle|\,A\subseteq C\right\}$. We claim that $S$ is closed in $K$, or equivalently that the complement $S^c=K\setminus S$ is open in $K$. Let $B\in S^c$, in other words, let $B$ be a non-empty compact set which is not contained in $C$. Choose $b\in B$ with $b\notin C$. Since $C$ is closed in $\mathbb{R}^2$, so that its complement $C^c=\mathbb{R}^2\setminus C$ is open in $\mathbb{R}^2$, we can choose $r>0$ so that $B_{\mathbb{R}^2}(b,r)\subseteq C^c$, where $B_{\mathbb{R}^2}(b,r)=\left\{x\in\mathbb{R}^2\,\middle|\,|x-b|<r\right\}$. We claim that $B_K(B,r)\subseteq S^c$, where $B_K(B,r)=\left\{A\in K\,\middle|\,D(A,B)<r\right\}$. Let $A\in K$ with $D(A,B)<r$. Since $D(A,B)<r$ we have $\max_{x\in B}d(x,A)<r$, so in particular we have $d(b,A)<r$. Since $d(b,A)<r$, we can choose $a\in A$ such that $d(b,a)<r$. Since $a\in B_{\mathbb{R}^2}(b,r)\subseteq C^c$ we have $a\in C^c$, that is $a\notin C$. Since $a\in A$ with $a\notin C$, we have $A\not\subseteq C$ so that $A\in S^c$. This shows that $B_K(B,R)\subseteq S^c$ so that $S^c$ is open in $K$, and hence $S$ is closed in $K$, as claimed.

**4:** (a) Let $K_n$ be the set of $x \in [0,1]$ which can be written in base 5 so that the first $n$ digits are not equal to 2 (that is the numbers of the form $x = \sum\limits_{k=1}^{\infty} \frac{x_k}{5^k}$ with $x_k \in \{0,1,3,4\}$ for $k \leq n$ and $x_k \in \{0,1,2,3,4\}$ for $k > n$) and let $C = \bigcap_{k=1}^{\infty} K_n$. Find the total length $L_n$ of each set $K_n$ and hence the total length $L = \lim\limits_{n\to\infty} L_n$ of $C$.

Solution: Let $K_0 = [0,1]$. We have $K_1 = \left[0, \frac{1}{5}\right] \cup \left[\frac{1}{5}, \frac{2}{5}\right] \cup \left[\frac{3}{5}, \frac{4}{5}\right] \cup \left[\frac{4}{5}, 1\right]$. The set $K_n$ is a union of $4^n$ closed intervals, which only overlap at endpoints, each of length $\frac{1}{5^n}$, and $K_{n+1}$ is obtained from $K_n$ by partitioning each of its interval of size $\frac{1}{5^n}$ into 5 equal closed subintervals of (each of size $\frac{1}{5^{n+1}}$) and removing the middle interval, leaving 4 intervals of size $\frac{1}{5^{n+1}}$ (two adjacent intervals on the left and two adjacent intervals on the right). To be explicit, $K_n$ is the union of the $4^n$ intervals $\left[c, c + \frac{1}{5^n}\right]$ where $c$ is of the form $c = \sum_{k=1}^{n} \frac{a_k}{5^k}$ with each $a_k \in \{0,1,3,4\}$. Since $K_n$ is a union of $4^n$ intervals of size $\frac{1}{5^n}$ (intersecting only at endpoints), the total length is $L_n = \left(\frac{4}{5}\right)^n$. The total length of $C$ is $\lim\limits_{n\to\infty} L_n = \lim\limits_{n\to\infty} \left(\frac{4}{5}\right)^n = 0$.

(b) Find the area inside the Koch snowflake (described in Example 7.21).

Solution: The region inside (and on) the Koch snowflake can be constructed as follows. We begin with an equilateral triangle $T_0$ of side length 1, with 3 edges. We attach 3 equilateral triangles of side-length $\frac{1}{3}$, one along the middle third of each edge of $T_0$, to obtain $T_1$, which is the union of 1 triangle of side-length 1 and 3 of side-length $\frac{1}{3}$ with $3 \cdot 4 = 12$ external edges each of length $\frac{1}{3}$. Having constructed $T_n$, which has $3 \cdot 4^n$ external edges of length $\frac{1}{3^n}$, we attach $3 \cdot 4^n$ triangles of side-length $\frac{1}{3^{n+1}}$, one along the middle third of each edge of $T_n$ to obtain $T_{n+1}$. Then the Koch snowflake is the set $K = \bigcup_{n=1}^{\infty} T_n$, which is the union of 1 triangle of side-length 1, 3 triangles of side-length $\frac{1}{3}$, $3 \cdot 4$ triangles of side-length $\frac{1}{3^2}$, and so on, with $3 \cdot 4^{n-1}$ triangles of side-length $\frac{1}{3^n}$ for each $n \geq 1$.

An equilateral triangle of side-length $\ell$ has area $\frac{1}{2} \cdot \ell \cdot \frac{\sqrt{3}}{2}\ell = \frac{\sqrt{3}}{4}\ell^2$, so the area of the region in the Koch curve is

$$\begin{aligned} A &= 1 \cdot \tfrac{\sqrt{3}}{4} + 3 \cdot \tfrac{\sqrt{3}}{4} \cdot \tfrac{1}{3^2} + 3 \cdot 4 \cdot \tfrac{\sqrt{3}}{4} \cdot \tfrac{1}{3^4} + 3 \cdot 4^2 \cdot \tfrac{\sqrt{3}}{4} \cdot \tfrac{1}{3^6} + 3 \cdot 4^3 \cdot \tfrac{\sqrt{3}}{4} \cdot \tfrac{1}{3^8} + \cdots \\ &= \tfrac{\sqrt{3}}{4} + \tfrac{\sqrt{3}}{12}\left(1 + \tfrac{4}{9} + \left(\tfrac{4}{9}\right)^2 + \left(\tfrac{4}{9}\right)^3 + \cdots\right) = \tfrac{\sqrt{3}}{4}\left(1 + \tfrac{1}{3} \cdot \tfrac{1}{1-\frac{4}{9}}\right) = \tfrac{\sqrt{3}}{4} \cdot \left(1 + \tfrac{3}{5}\right) = \tfrac{2\sqrt{3}}{5}. \end{aligned}$$

**5:** (a) Find the exact similarity dimension of the self-similar set $R$ shown below in blue.

Solution: $R$ is equal to the union of $m = 3$ copies of itself scaled by $s = \frac{1}{2}$, so $d_{\text{sim}}(R) = \frac{\ln m}{\ln \frac{1}{s}} = \frac{\ln 3}{\ln 2}$.

(b) Find the exact similarity dimension of the self-similar shape $S$ shown below in brown, and find formulas for similarities $F_1$, $F_2$ and $F_3$ such that $S = F_1(S) \cup F_2(S) \cup F_3(S)$.

Solution: We can, for example, take $F_1\binom{x}{y} = \frac{1}{4}\binom{x}{y}$, $F_2\binom{x}{y} = \frac{1}{2}\binom{x}{y} + \binom{1/2}{0}$ and $F_3\binom{x}{y} = \frac{1}{2}\binom{x}{y} + \binom{1/4}{\sqrt{3}/4}$. There are other possible choices, for example we could reflect in the line $y = x$ and take $F_1\binom{x}{y} = \frac{1}{4}\binom{y}{x}$. Since there is one similarity of scaling factor $\frac{1}{4}$ and two of scaling factor $\frac{1}{2}$, the similarity dimension $d = d_{\text{sim}}(S)$ is given by $\left(\frac{1}{4}\right)^d + 2\left(\frac{1}{2}\right)^d = 1$. Letting $u = \left(\frac{1}{2}\right)^d$, we have $u^2 + 2u - 1 = 0$, and so $u = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$. Since $u > 0$ we have $u = -1 + \sqrt{2}$, that is $\left(\frac{1}{2}\right)^d = \sqrt{2} - 1$, so $2^d = \frac{1}{\sqrt{2}-1} = 1 + \sqrt{2}$, hence $d = \frac{\ln(1+\sqrt{2})}{\ln 2}$.

(c) Find the exact similarity dimension of $T$, and find the exact coordinates of 3 points which lie in $T$, where $T$ is the self-similar shape shown below in green, with $T = G_1(T) \cup G_2(T) \cup G_3(T)$ where $G_1$, $G_2$ and $G_3$ are given by

$$G_1\begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{50}\begin{pmatrix} 9\,x + 9\sqrt{3}\,y + 25\sqrt{3} \\ -9\sqrt{3}\,x + 9\,y + 25 \end{pmatrix},\ G_2\begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{50}\begin{pmatrix} 9\,x - 9\sqrt{3}\,y - 25\sqrt{3} \\ 9\sqrt{3}\,x + 9\,y + 25 \end{pmatrix},\ G_3\begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{3}{5}\begin{pmatrix} x \\ y+5 \end{pmatrix}.$$

Solution: Note that $G_1\binom{x}{y} = \frac{9}{25}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\binom{x}{y} + \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$ and $G_2\binom{x}{y} = \frac{9}{25}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\binom{x}{y} + \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$ (and the two matrices are both rotation matrices), so the maps $G_1$ and $G_2$ both scale by $\frac{9}{25}$, and $G_3$ scales by $\frac{3}{5}$. The similarity dimension $d = d_{\text{sim}}(T)$ is given by $2\left(\frac{9}{25}\right)^d + \left(\frac{3}{5}\right)^d = 1$. Letting $u = \left(\frac{3}{5}\right)^d$, we have $0 = 2u^2 + u - 1 = (2u-1)(u+1)$ so that $u = \frac{1}{2}$ (since $u > 0$). Since $\left(\frac{3}{5}\right)^d = \frac{1}{2}$ we have $d = \frac{\ln \frac{1}{2}}{\ln \frac{3}{5}} = \frac{\ln 2}{\ln \frac{5}{3}}$.

Recall that for any $a \in \mathbb{R}^2$ we have $F^n(\{a\}) \to T$ (in the space of nonempty compact sets using the Hausdorff metric). If $a$ is a fixed point of one of the maps $F_k$ then $a \in F^n(\{a\})$ for all $n$, so that $a$ is in the limiting set $T$. We have $G_3\binom{x}{y} = \binom{x}{y} \iff \left(\frac{3}{5}x = x \text{ and } \frac{3}{5}y + 3 = y\right) \iff \binom{x}{y} = \binom{0}{\frac{15}{2}}$, and hence $\binom{0}{\frac{15}{2}} \in T$. We also have $G_1\binom{0}{\frac{15}{2}} = \frac{9}{25}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\binom{0}{\frac{15}{2}} + \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} = \frac{1}{100}\begin{pmatrix} 185\sqrt{3} \\ 185 \end{pmatrix} = \frac{37}{20}\binom{\sqrt{3}}{1} \in T$ and $G_2\binom{0}{\frac{15}{2}} = \frac{37}{20}\binom{-\sqrt{3}}{1} \in T$