**1:** (a) Define $f:\mathbb{R}^2\to\mathbb{R}^2$ by $f(x,y)=\left(x,\frac{y}{2}\right)$. Show that $d\big(f(u),f(v)\big)\le d(u,v)$ for all $u,v\in\mathbb{R}^2$, determine whether $f$ is a contraction map, and determine whether $f$ has a unique fixed point in $\mathbb{R}^2$.

(b) The polynomial $p(x)=x^3-3x+1$ has a unique root in $\left[0,\frac{1}{2}\right]$. Approximate this root using the Banach Fixed Point Theorem as follows: Let $f(x)=\frac{1}{3}(x^3+1)$. Show that $f:\left[0,\frac{1}{2}\right]\to\left[0,\frac{1}{2}\right]$ is a contraction map whose unique fixed point is the desired root of $p$. Approximate the root by using a calculator to find $x_5$ where $x_0=0$ and $x_{n+1}=f(x_n)$.

**2:** (a) Define $F:\mathbb{R}^2\to\mathbb{R}$ by $F(x,y)=3\,y^{2/3}$. Determine whether $F$ satisfies the hypothesis of Picard's Theorem, whether $F$ satisfies the hypothesis of Peano's Theorem, and whether there exists $\delta>0$ such that the differential equation $\frac{dy}{dx}=F(x,y)$ has a unique solution $y=f(x)$ with $f(0)=0$, defined for all $x\in(-\delta,\delta)$.

(b) Let $a_1,a_2,\cdots,a_n\in\mathbb{R}^n$ with say $a_k=(a_{k,1},a_{k,2},\cdots,a_{k,n})$, and let $A\in M_n(\mathbb{R})$ be that matrix with entries $a_{k,\ell}$. By applying the Banach Fixed Point Theorem to the map $F:(\mathbb{R}^n,d_\infty)\to(\mathbb{R}^n,d_\infty)$ given by $F(x)=Ax+b$, where $b\in\mathbb{R}^n$, show that if $\|a_k\|_1<1$ for all indices $k$ then the matrix $I-A$ is invertible.

**3:** (a) Find $D(A,B)$ where $D$ is the Hausdorff metric and $A$ is the line segment in $\mathbb{R}^2$ from $(0,0)$ to $(3,4)$ and $B$ is the line segment in $\mathbb{R}^2$ from $(0,1)$ to $(4,3)$.

(b) Let $K$ be the set of all nonempty compact sets in $\mathbb{R}^2$. Show that for every closed set $C\subseteq\mathbb{R}^2$ (in the standard metric), the set $S=\left\{A\in K\,\middle|\,A\subseteq C\right\}$ is closed in $K$, using the Hausdorff metric.

**4:** (a) Let $K_n$ be the set of $x\in[0,1]$ which can be written in base 5 so that the first $n$ digits are not equal to 2 (that is the numbers of the form $x=\sum\limits_{k=1}^{\infty}\frac{x_k}{5^k}$ with $x_k\in\{0,1,3,4\}$ for $k\le n$ and $x_k\in\{0,1,2,3,4\}$ for $k>n$) and let $C=\bigcap_{k=1}^{\infty}K_n$. Find the total length $L_n$ of each set $K_n$ and hence the total length $L=\lim\limits_{n\to\infty}L_n$ of $C$.

(b) Find the area inside the Koch snowflake (constructed starting with an equilateral triangle with unit sides).

**5:** (a) Find the exact similarity dimension of the self-similar set $R$ shown below in blue.

(b) Find the exact similarity dimension of the self-similar shape $S$ shown below in brown, and find formulas for similarities $F_1$, $F_2$ and $F_3$ such that $S=F_1(S)\cup F_2(S)\cup F_3(S)$.

(c) Find the exact similarity dimension of $T$, and find the exact coordinates of 3 points which lie in $T$, where $T$ is the self-similar shape shown below in green, with $T=G_1(T)\cup G_2(T)\cup G_3(T)$ where $G_1$, $G_2$ and $G_3$ are given by

$$G_1\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{50}\begin{pmatrix}9\,x+9\sqrt{3}\,y+25\sqrt{3}\\-9\sqrt{3}\,x+9\,y+25\end{pmatrix}\ ,\ G_2\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{50}\begin{pmatrix}9\,x-9\sqrt{3}\,y-25\sqrt{3}\\9\sqrt{3}\,x+9\,y+25\end{pmatrix}\ ,\ G_3\begin{pmatrix}x\\y\end{pmatrix}=\frac{3}{5}\begin{pmatrix}x\\y+5\end{pmatrix}.$$