AMATH/PMATH 331 Solutions to Assignment 5

1: For each of the following sets A in $\mathbb{R}^n$ (using its standard metric), determine whether A is complete and whether A is compact.

(a) $A = \left\{(a,b,c,d) \in \mathbb{R}^4 \,\middle|\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right\}$.

Solution: We have $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{smallmatrix}\right)$ and hence

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \iff \big(a^2+bc=a\,,\ ab+bd=b\,,\ ac+cd=d \text{ and } d^2+bc=d\big).$$

The set A is closed because each of the four functions $f_k : \mathbb{R}^4 \to \mathbb{R}$ given by $f_1(a,b,c,d) = a^2+bc-a$, $f_2(a,b,c,d) = ab+bd-b$, $f_3(a,b,c,d) = ac+cd-d$ and $f_4(a,b,c,d) = d^2+bc-d$ is continuous, so that each of the sets four $f_k^{-1}(0)$ is closed, and we have $A = \bigcap_{k=1}^4 f_k^{-1}(0)$. Since A is closed, it is complete. Note that A is unbounded because for every $0 < t \in \mathbb{R}$ we have $\frac{1}{2}\big(1, t, \frac{1}{t}, 1\big) \in A$. Since A is unbounded it is not compact.

(b) $A = \left\{x \in \mathbb{R}^3 \,\middle|\, \|x\| = \frac{n-1}{n^2} \text{ for some } n \in \mathbb{Z}^+\right\}$.

Solution: Let $(r_n)_{n\geq 1}$ be the sequence given by $r_n = \frac{n-1}{n^2}$ and let $R = \{r_n \,|\, n \in \mathbb{Z}^+\}$. Note that $r_1 = 0$, and $r_2 = \frac{1}{4}$, and r_n is strictly decreasing for $n \geq 2$ because for $f(x) = \frac{x-1}{x^2}$ we have $f'(x) = \frac{x^2-2x(x-1)}{x^4} = \frac{x(2-x)}{x^4}$ so that $f'(2) = 0$ and $f'(x) < 0$ for $x > 2$. Because $r_1 = 0$ and $r_2 = \frac{1}{4}$ and (r_n) is decreasing for $n \geq 2$, it follows that $\max R = r_2 = \frac{1}{4}$, and hence A is bounded with $A \subseteq \overline{B}\big(0, \frac{1}{4}\big)$ $\big($because when $x \in A$ we have $\|x\| \in R$ so that $\|x\| \leq \max R = \frac{1}{4}\big)$.

We claim that A is closed. Because $r_1 = 0$ and $r_2 = \frac{1}{4}$ and (r_n) is strictly decreasing for $n \geq 2$ with $\lim_{n\to\infty} r_n = 0$, it follows that $[0,\infty) \setminus R = \bigcup_{n=1}^\infty I_n$ where $I_1 = \big(\frac{1}{4}, \infty\big)$ and $I_n = (r_{n+1}, r_n)$ for $n \geq 2$. For each $n \in \mathbb{Z}^+$, let $U_n = \big\{x \in \mathbb{R}^3 \,\big|\, \|x\| \in I_n\big\}$, and note that each of the sets U_n is open in $\mathbb{R}^3$: indeed $U_1 = \{x \in \mathbb{R}^3 \,|\, \|x\| > \frac{1}{4}\} = \overline{B}\big(0, \frac{1}{4}\big)^c$, which is open, and for $n \geq 2$ we have $U_n = \big\{x \in \mathbb{R}^3 \,\big|\, r_{n+1} < \|x\| < r\big\} = B\big(0, r_n\big) \cap \overline{B}(0, r_{n+1})^c$, which is open. Finally note that for $x \in \mathbb{R}^3$, we have

$$A^c = \big\{x \in \mathbb{R}^3 \,\big|\, \|x\| \notin R\big\} = \big\{x \in \mathbb{R}^3 \,\big|\, \|x\| \in \textstyle\bigcup_{n=1}^\infty I_n\big\} = \textstyle\bigcup_{n=1}^\infty U_n$$

which is open in $\mathbb{R}^3$ (since it is a union of open sets). Since A^c is open, it follows that A is closed, as claimed. Since A is closed, it is complete, and since A is closed and bounded, it is compact.

2: For each of the following sets A, determine whether A is complete and whether A is compact.

(a) $A = \left\{ x \in \mathbb{R}^\infty \,\middle|\, \|x\|_\infty \leq 1 \right\} \subseteq \mathbb{R}^\infty \subseteq \ell_\infty$, using the metric d_∞.

Solution: We claim that A is not closed in ℓ_∞ (using the metric d_∞). Let $(x_n)_{n\geq 1}$ be the sequence in $\mathbb{R}^\infty$ given by $x_n = \sum\limits_{k=1}^{n} \frac{1}{k}\, e_k = \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, 0, 0, \cdots\right)$. Note that for every $n \in \mathbb{Z}^+$ we have $\|x_n\|_\infty = 1$ so that $x_n \in A$, and so $(x_n)_{n\geq 1}$ is a sequence in A. Also note that $x_n \to a$ in ℓ_∞ where $a = (a_k)_{k\geq 1} \in \ell_\infty$ is given by $a_k = \frac{1}{k}$ for all $k \in \mathbb{Z}^+$, that is $a = \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots\right)$, indeed we have

$$\big\|a - x_n\big\|_\infty = \Big\|\left(0, 0, \cdots, 0, \tfrac{1}{n+1}, \tfrac{1}{n+2}, \tfrac{1}{n+3}, \cdots\right)\Big\|_\infty = \tfrac{1}{n+1} \longrightarrow 0 \text{ as } n \to \infty.$$

But notice that $a \notin \mathbb{R}^\infty$, so $a \notin A$. Since $(x_n)_{\geq 1}$ is a sequence in A with $x_n \to a$ in ℓ_∞, but $a \notin A$, it follows (from Theorem 5.16) that A is not closed in ℓ_∞. Since ℓ_∞ is complete and A is not closed in ℓ_∞, it follows (from Theorem 6.4) that A is not complete. Since A is not closed in ℓ_∞, it follows (from Theorem 6.21) that A is not compact in ℓ_∞ (and we remark that, by Theorem 6.19, A is not compact in $\mathbb{R}^\infty$ or in itself).

(b) $A = \left\{ x \in \ell_\infty \,\middle|\, \|x\|_2 \leq 1 \right\}$ in the metric space $\left(\ell_\infty, d_\infty\right)$.

Solution: We claim that A is closed in (ℓ_∞, d_∞), and hence A is complete (since (ℓ_∞, d_∞) is complete). Let $a \in A^c$. We have $\|a\|_2 > 1$, and so $\sum_{k=1}^\infty |a_k|^2 > 1$. Choose $n \in \mathbb{Z}^+$ such that $\sum_{k=1}^n |a_k|^2 > 1$. Let $b = (a_1, \cdots, a_n) \in \mathbb{R}^n$ and note that $\|b\|_2 = \left(\sum_{k=1}^n |a_k|^2\right)^{1/2} > 1$. Let $r = \frac{\|b\|_2 - 1}{\sqrt{n}}$. Let $x \in B_\infty(a, r)$. Let $y = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and note that $\|y - b\|_\infty \leq \|x - a\|_\infty < r$. Recall that for $u \in \mathbb{R}^n$ we have

$$\|u\|_2 = \left(\textstyle\sum_{k=1}^n |u_k|^2\right)^{1/2} \leq \left(\textstyle\sum_{k=1}^n \|u\|_\infty^2\right)^{1/2} = \left(n\|u\|_\infty^2\right)^{1/2} = \sqrt{n}\,\|u\|_\infty.$$

Using the (reverse) Triangle Inequality in $\mathbb{R}^n$, we have

$$\|x\|_2 \geq \|y\|_2 \geq \|b\|_2 - \|y - b\|_2 \geq \|b\|_2 - \sqrt{n}\,\|y - b\|_\infty > \|b\|_2 - \sqrt{n}\,r = 1$$

so that $x \in A^c$. Thus $B_\infty(a, r) \subseteq A^c$, so that A is closed in (ℓ_∞, d_∞) (and hence complete), as claimed.

Finally note that A is not compact since, letting e_n denote the n^{th} standard basis vector in $\mathbb{R}^\infty$, the sequence $(e_n)_{n\geq 1}$ is a sequence in A with no convergent subsequence (when $n \neq m$ we have $\|e_n - e_m\|_\infty = 1$).

3: (a) Let $A = \left\{ f \in \mathcal{C}[0,1] \,\middle|\, |f(x)| \leq \frac{1}{x} \text{ for all } x \in (0,1) \right\} \subseteq \mathcal{C}[0,1]$. determine whether A is complete and whether A is compact using the metric d_∞.

Solution: We claim that A is closed in $\mathcal{C}[0,1]$ (using the supremum metric d_∞). Let $f \in A^c = \mathcal{C}[0,1] \setminus A$. Choose $a \in (0,1)$ such that $\big|f(a)\big| > \frac{1}{a}$. Let $r = \big|f(a)\big| - \frac{1}{a}$ and note that $r > 0$. We claim that $B(f,r) \subseteq A^c$. Let $g \in B(f,r)$, that is let $g \in \mathcal{C}[0,1]$ with $\|g - f\|_\infty \leq 1$. Since $|f(a)| \leq \big|f(a) - g(a)\big| + \big|g(a)\big|$, we have

$$|f(a)| - |g(a)| \leq \big|f(a) - g(a)\big| \leq \|f - g\|_\infty < r = \big|f(a)\big| - \tfrac{1}{a}$$

and hence $\big|g(a)\big| > \frac{1}{a}$, so that $g \in A^c$. Thus $B(f,r) \subseteq A^c$, showing that A^c is open, hence A is closed, as claimed. Since $\mathcal{C}[0,1]$ is complete and A is closed in $\mathcal{C}[0,1]$, it follows that A is complete.

On the other hand, A is not bounded because given $r > 0$ we can define $f : [0,1] \to \mathbb{R}$ by $f(x) = r$ for $0 \leq x \leq \frac{1}{r}$ and $f(x) = \frac{1}{x}$ for $\frac{1}{r} \leq x \leq 1$ and then we have $f \in A$ with $\|f\|_\infty = r$. Since A is not bounded, it is not compact.

(b) Define $F : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ by $F(f) = f^2$, that is by $F(f)(x) = f(x)^2$ for all $x \in [0,1]$ (note that F is not linear). Determine whether F is continuous as a map $F : \big(\mathcal{C}[0,1], d_1\big) \to \big(\mathcal{C}[0,1], d_2\big)$ and whether F is continuous as a map $F : \big(\mathcal{C}[0,1], d_2\big) \to \big(\mathcal{C}[0,1], d_1\big)$.

Solution: We claim that $F : (\mathcal{C}[0,1], d_1) \to (\mathcal{C}[0,1], d_2)$ is not continuous at 0. We need to show that there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $f \in \mathcal{C}[0,1]$ such that $\|f\|_1 < \delta$ but $\|f^2\|_2 \geq \epsilon$. Take $\epsilon = 1$. For $n \in \mathbb{Z}^+$, let $f_n(x) = n\,\delta\,x^n$. Then $\|f_n\|_1 = \frac{n\,\delta}{n+1} < \delta$, and $\|{f_n}^2\|_2 = \frac{n\,\delta}{\sqrt{2n+1}} \to \infty$ so that we can choose $n \in \mathbb{Z}^+$ such that $\|{f_n}^2\|_2 \geq 1 = \epsilon$.

We claim that $F : (\mathcal{C}[0,1], d_2) \to (\mathcal{C}[0,1], d_1)$ is continuous (at every $g \in \mathcal{C}[0,1]$). Let $g \in \mathcal{C}[0,1]$. Let $\epsilon > 0$. Choose $\delta = \min\left\{1, \frac{\epsilon}{2M+1}\right\}$ where $M = \|g\|_2$. Let $f \in \mathcal{C}[0,1]$ with $\|f - g\|_2 < \delta$. Note that since $\|f - g\|_2 < \delta \leq 1$ we have $\|f\|_2 \leq \|f - g\|_2 + \|g\|_2 < 1 + M$ and so, using the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} \|f^2 - g^2\|_1 &= \int_0^1 |f - g|\,|f + g| = \big\langle |f - g|, |f + g| \big\rangle \leq \|f - g\|_2 \|f + g\|_2 \\ &\leq \|f - g\|_2 \big(\|f\|_2 + \|g\|_2\big) < \tfrac{\epsilon}{2M+1}\big((1+M) + M\big) = \epsilon. \end{aligned}$$

4: (a) Let X be a compact metric space. Let $(A_n)_{n\geq 1}$ be a sequence of nonempty closed subsets of X with $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$. Prove that $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

Solution: Suppose, for a contradiction, that $\bigcap_{n=1}^{\infty} A_n = \emptyset$. By taking the complement, we have $\bigcup_{n=1}^{\infty} A_n^c = X$ where $A_n^c = X \setminus A_n$, so that $\left\{A_n^c \,\middle|\, n \in \mathbb{Z}^+\right\}$ is an open cover of X. Since X is compact, we can choose a finite subcover, so we can choose $\ell \in \mathbb{Z}^+$ such that $\bigcup_{n=1}^{\ell} A_n^c = X$. By taking the complement, we have $\bigcap_{n=1}^{\ell} A_n = \emptyset$. This is not possible since $A_1 \supseteq A_2 \supseteq \cdots$ so that $\bigcap_{n=1}^{\ell} A_n = A_\ell$, and $A_\ell \neq \emptyset$.

(b) Let X be a metric space and let $A, B \subseteq X$. Show that if A is compact, B is closed and $A \cap B = \emptyset$, then there exists $r > 0$ such that the open sets $U = \bigcup_{a\in A} B(a, r)$ and $V = \bigcup_{b\in B} B(b, r)$ are disjoint.

Solution: Suppose that A is compact, B is closed and $A \cap B = \emptyset$. Since $A \cap B = \emptyset$ we have $A \subseteq B^c$. For each $a \in A$, since $a \in B^c$ and B^c is open, we can choose $d_a > 0$ such that $B(a, 2d_a) \subseteq B^c$. Note that the set $S = \left\{B(a, d_a) \,\middle|\, a \in A\right\}$ is an open cover of A. Since A is compact, we can choose a finite subcover of S, so we can choose $a_1, a_2, \cdots, a_n \in A$ such that $A \subseteq B(a_1, d_{a_1}) \cup \cdots \cup B(a_n, d_{a_n})$. Let $d = \min\{d_{a_1}, \cdots, d_{a_n}\}$.

We claim that $d(a, b) \geq d$ for all $a \in A$ and $b \in B$. Suppose, for a contradiction, that $a \in A$ and $b \in B$ and $d(a, b) < d$. Since $a \in A \subseteq B(a_1, d_{a_1}) \cup \cdots \cup B(a_n, d_{a_n})$ we can choose k such that $a \in B(a_k, d_{a_k})$. Then we have $d(a, a_k) < d_{a_k}$ and $d(a, b) < d \leq d_{a_k}$ hence $d(b, a_k) \leq d(b, a) + d(a, a_k) < 2d_{a_k}$ so that $b \in B(a_k, 2d_{a_k})$. This is not possible since $B(a_k, 2d_{a_k}) \subseteq B^c$. Thus $d(a, b) \geq d$ for all $a \in A$ and $b \in B$, as claimed.

Let $r = \frac{d}{2}$, and let $U = \bigcup_{a\in A} B\big(a, r\big)$ and $V = \bigcup_{b\in B} B\big(b, r\big)$. Then U and V are disjoint because if we had $x \in U \cap V$ then, since $x \in U$ we could choose $a \in A$ such that $x \in B\big(a, r\big)$, and since $x \in V$ we could choose $b \in B$ so that $x \in B\big(b, r\big)$, but then we would have $d(a, b) \leq d(a, x) + d(x, b) < 2r = d$.