1: For each of the following sets A in $\mathbb{R}^n$ (using its standard metric), determine whether A is complete and whether A is compact.

(a) $A = \left\{ (a,b,c,d) \in \mathbb{R}^4 \,\middle|\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \right\}$.

(b) $A = \left\{ x \in \mathbb{R}^3 \,\middle|\, \|x\| = \frac{n-1}{n^2} \text{ for some } n \in \mathbb{Z}^+ \right\}$.

2: For each of the following sets A, determine whether A is complete and whether A is compact.

(a) $A = \left\{ x \in \mathbb{R}^\infty \,\middle|\, \|x\|_\infty \leq 1 \right\} \subseteq \mathbb{R}^\infty \subseteq \ell_\infty$, using the metric d_∞.

(b) $A = \left\{ x \in \ell_\infty \,\middle|\, \|x\|_2 \leq 1 \right\}$ in the metric space $\big(\ell_\infty, d_\infty\big)$.

3: (a) Let $A = \left\{ f \in \mathcal{C}[0,1] \,\middle|\, |f(x)| \leq \frac{1}{x} \text{ for all } x \in (0,1) \right\} \subseteq \mathcal{C}[0,1]$. determine whether A is complete and whether A is compact using the metric d_∞.

(b) Define $F : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ by $F(f) = f^2$, that is by $F(f)(x) = f(x)^2$ for all $x \in [0,1]$ (note that F is not linear). Determine whether F is continuous as a map $F : \big(\mathcal{C}[0,1], d_1\big) \to \big(\mathcal{C}[0,1], d_2\big)$ and whether F is continuous as a map $F : \big(\mathcal{C}[0,1], d_2\big) \to \big(\mathcal{C}[0,1], d_1\big)$.

4: (a) Let X be a compact metric space. Let $(A_n)_{n\geq 1}$ be a sequence of nonempty closed subsets of X with $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$. Prove that $\bigcap_{n=1}^\infty A_n \neq \emptyset$.

(b) Let X be a metric space and let $A, B \subseteq X$. Show that if A is compact, B is closed and $A \cap B = \emptyset$, then there exists $r > 0$ such that the open sets $U = \bigcup_{a\in A} B(a,r)$ and $V = \bigcup_{b \in B} B(b,r)$ are disjoint.