AMATH/PMATH 331 Solutions to Assignment 4

1: (a) Let $A=\{(x,y)\in\mathbb{R}^2\,|\,4x^2+y^2<8x\}$. Prove that A is open in $\mathbb{R}^2$.

Solution: We have $A=f^{-1}\big((-\infty,0)\big)$ where $f:\mathbb{R}^2\to\mathbb{R}$ is given by $f(x,y)=4x^2+y^2-8x$. Since $(-\infty,0)$ is open and f is continuous, it follows that A is open (by the topological characterization of continuity).

(b) Let $B=\left\{(a,b,c,d)\in\mathbb{R}^4\,\middle|\,\begin{pmatrix}a&b\\c&d\end{pmatrix}^2=\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}$. Prove that B is closed in $\mathbb{R}^4$.

Solution: For $a,b,c,d\in\mathbb{R}$ we have $\begin{pmatrix}a&b\\c&d\end{pmatrix}^2=\begin{pmatrix}a^2+bc&ab+bd\\ac+cd&bc+d^2\end{pmatrix}$ so that

$$(a,b,c,d)\in B\iff\big(a^2+bc\,,\,ab+bd\,,\,ac+cd\,,\,bc+d^2\big)=(1,0,0,1),$$

and hence $B=g^{-1}(p)$ where $g:\mathbb{R}^4\to\mathbb{R}^4$ is given by $g(a,b,c,d)=\big(a^2+bc,ab+bd,ac+cd,bc+d^2\big)$ and $p=(1,0,0,1)\in\mathbb{R}^4$. The map g is continuous (it is a polynomial map) and $\{p\}$ is closed in $\mathbb{R}^4$, and so the set $B=g^{-1}\big(\{p\}\big)$ is closed in $\mathbb{R}^4$ (by the topological characterization of continuity).

(c) Let $C=\{(t^2-1\,,\,t^3-t)\in\mathbb{R}^2\,|\,t\in\mathbb{R}\}$. Determine whether C is closed in $\mathbb{R}^2$.

Solution: We claim that $C=h^{-1}(0)$ where $h:\mathbb{R}^2\to\mathbb{R}$ is given by $h(x,y)=x^3+x^2-y^2$. Let $(x,y)\in C$, say $(x,y)=(t^2-1\,,\,t^3-t)$. Then $x^3+x^2=(t^6-3t^4+3t^2-1)+(t^4-2t^2+1)=t^6-2t^4+t^2=(t^3-t)^2=y^2$ so that $h(x,y)=0$. This shows that $C\subseteq h^{-1}(0)$. Now let $(x,y)\in h^{-1}(0)$, so we have $y^2=x^3+x^2$. If $x=0$ then $y^2=x^3+x^2=0$ so that $y=0$, and in this case we can choose $t=1$ to get $t^2-1=0=x$ and $t^3-t=0=y$ so that $(x,y)\in C$. If $x\neq 0$ then we can choose $t=\frac{y}{x}$ to get $t^2-1=\frac{y^2}{x^2}-1=\frac{y^2-x^2}{x^2}=\frac{x^3}{x^2}=x$ and $t^3-t=t(t^2-1)=\frac{y}{x}\cdot x=y$ so that again $(x,y)\in C$. This shows that $h^{-1}(0)\subseteq C$, and hence $C=h^{-1}(0)$, as claimed. Since $\{0\}$ is closed and h is continuous, it follows (from the topological characterization of continuity) that C is closed.

2: We consider that $\mathbb{C} = \mathbb{R}^2$ (when $x, y \in \mathbb{R}$, the ordered pair $(x, y) \in \mathbb{R}$ is equal to the complex number $z = x + iy \in \mathbb{C}$), and the usual norm in $\mathbb{C}$ is equal to the usual norm in $\mathbb{R}^2$: for $z = x + iy = (x, y)$ we have $\|z\| = \sqrt{x^2 + y^2}$. Recall that for $z, w \in \mathbb{C}$ we have $\|zw\| = \|z\|\,\|w\|$.

(a) For $n \geq 1$, let $s_n = \sum\limits_{k=1}^{n} \left(\frac{1+i}{3}\right)^k$. Prove, from the definition of a limit, that $\lim\limits_{n\to\infty} s_n = \frac{1+3\,i}{5}$ in $\mathbb{C}$.

Solution: From the formula for the sum of a geometric series, or by noting that

$$s_n\left(1 - \tfrac{1+i}{3}\right) = \sum_{k=1}^{n} \left(\tfrac{1+i}{3}\right)^k - \sum_{k=2}^{n+1} \left(\tfrac{1+i}{3}\right)^k = \left(\tfrac{1+i}{3}\right) - \left(\tfrac{1+i}{3}\right)^{n+1},$$

we have

$$s_n = \frac{(\frac{1+i}{3}) - (\frac{1+i}{3})^{n+1}}{1 - \frac{1+i}{3}} = \frac{(\frac{1+i}{3})(1 - (\frac{1+i}{3})^n)}{\frac{2-i}{3}} = \frac{(1+i)(2+i)(1 - (\frac{1+i}{3})^n)}{(2-i)(2+i)} = \tfrac{1+3i}{5}\left(1 - \left(\tfrac{1+i}{3}\right)^n\right) = \tfrac{1+3i}{5} - \tfrac{1+3i}{5}\left(\tfrac{1+i}{3}\right)^n$$

and hence

$$\left\|s_n - \tfrac{1+3i}{5}\right\| = \left\|\tfrac{1+3i}{5}\left(\tfrac{1+i}{3}\right)^n\right\| = \left|\tfrac{1+3i}{5}\right| \left\|\tfrac{1+i}{3}\right\|^n = \tfrac{\sqrt{10}}{5}\left(\tfrac{\sqrt{2}}{3}\right)^n.$$

Using the definition of the limit, it follows that $\lim\limits_{n\to\infty} s_n = \frac{1+3i}{5}$: indeed given $\epsilon > 0$, since $\frac{\sqrt{2}}{3} < 1$ we can choose $m \in \mathbb{Z}^+$ so that $\left(\frac{\sqrt{2}}{3}\right)^m < \frac{\epsilon}{\sqrt{10}/5}$, and then when $n \geq m$ we have

$$\left\|s_n - \tfrac{1+3i}{5}\right\| = \tfrac{\sqrt{10}}{5}\left(\tfrac{\sqrt{2}}{3}\right)^n \leq \tfrac{\sqrt{10}}{5}\left(\tfrac{\sqrt{3}}{2}\right)^m < \epsilon.$$

(b) Define $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ by $f(z) = \frac{z^2 - \overline{z}^2}{\|z\|^2}$. Prove, from the definition of a limit, that $\lim\limits_{z\to 0} f(z)$ does not exist.

Solution: Note that $f(x + iy) = \frac{((x^2-y^2)+i\,2xy) - ((x^2+y^2)-i\,2xy)}{x^2+y^2} = \frac{i\,4xy}{x^2+y^2}$, and we have $f(x + i0) = 0$ and $f(x, x) = \frac{i\,4x^2}{2x^2} = 2i$. We use this to prove, from the definition of the limit, that $\lim_{z\to 0} f(z)$ cannot exist. Suppose, for a contradiction, that $\lim_{z\to 0} f(z)$ does exist, and let $b = \lim_{z\to 0} f(z)$. Taking $\epsilon = 1$, we can choose $\delta > 0$ such that for all $z \neq 0$, if $0 < \|z\| < \delta$ then $\left\|f(z) - b\right\| < 1$. When $z = \frac{\delta}{2}$ we have $0 < \|z\| = \frac{\delta}{2} < \delta$ and we have $f(z) = 0$, and hence $\left|0 - b\right| < 1$ so that $b \in B(0, 1)$. On the other hand, when $z = \frac{\delta}{2}(1 + i)$, we have $0 < \|z\| = \frac{\delta}{\sqrt{2}} < \delta$ and we have $f(z) = 2i$, and hence $\left|2i - b\right| < 1$, so that $b \in B(2i, 1)$. This gives the desired contradiction, since $B(0, 1) \cap B(2i, 1) = \emptyset$ (if we had $b \in B(0, 1)$ and $b \in B(2i, 1)$ then we would have $d(b, 0) < 1$ and $d(b, 2i) < 1$ but then, by the Triangle Inequality, we would have $d(0, 2i) \leq d(0, b) + d(b, 2i) < 1 + 1 = 2$).

3: (a) Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = 1 - nx$ for $0 \le x \le \frac{1}{n}$ and $f_n(x) = 0$ for $\frac{1}{n} \le x \le 1$. Show that $f_n \to 0$ in $\mathcal{C}[0,1]$ using either of the metrics d_1 or d_2, but $f_n \not\to 0$ pointwise on $[0,1]$.

Solution: We have $f_n \to 0$ in $\big(\mathcal{C}[0,1], d_1\big)$ and $f_n \to 0$ in $\big(\mathcal{C}[0,1], d_2\big)$ by Part 5 of Theorem 5.2 because

$$d_1(f_n, 0) = \int_0^1 |f_n(x)|\,dx = \int_0^{1/n} 1 - nx\;dx = \Big[x - \tfrac{n}{2}\,x^2\Big]_0^{1/n} = \tfrac{1}{2n} \to 0\ , \text{ and}$$

$$d_2(f_n, 0)^2 = \int_0^1 f_n(x)^2\,dx = \int_0^{1/n} 1 - 2nx + n^2x^2\;dx = \Big[x - nx^2 + \tfrac{n^2}{3}\,x^3\Big]_0^{1/n} = \tfrac{1}{3n} \to 0.$$

On the other hand, it is not the case that $f_n \to 0$ pointwise on $[0,1]$ because $\lim\limits_{n\to\infty} f_n(1) = \lim\limits_{n\to\infty} 1 = 1$.

(b) Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = n^2x - n^3x^2$ for $0 \le x \le \frac{1}{n}$ and $f_n(x) = 0$ for $\frac{1}{n} \le x \le 1$. Show that $f_n \to 0$ pointwise on $[0,1]$ but $f_n \not\to 0$ in $\mathcal{C}[0,1]$ using either of the metrics d_1 or d_2.

Solution: We claim that $f_n \to 0$ pointwise on $[0,1]$. When $x = 0$ we have $f_n(x) = f_n(0) = 0$ for all $n \in \mathbb{Z}^+$ so that $\lim\limits_{n\to\infty} f_n(x) = 0$. Let $x \in (0,1]$. Choose $m \in \mathbb{Z}^+$ large enough so that $\frac{1}{m} < x$. Then for $n \ge m$ we have $\frac{1}{n} \le \frac{1}{m} < x$. so that $f_n(x) = 0$. Since $f_n(x) = 0$ for all $n \ge m$, we have $\lim\limits_{n\to\infty} f_n(x) = 0$. Thus $f_n \to 0$ pointwise on $[0,1]$, as claimed.

On the other hand, we have $f_n \not\to 0$ in $\big(\mathcal{C}[0,1], d_1\big)$ and $f_n \not\to 0$ in $\big(\mathcal{C}[0,1], d_2\big)$ by Part 5 of Theorem 5.2 because

$$d_1(f_n, 0) = \int_0^1 |f_n(x)|\,dx = \int_0^{1/n} n^2x - n^3x^2\;dx = \Big[\tfrac{n^2}{2}x^2 - \tfrac{n^3}{3}x^3\Big]_0^{1/n} = \tfrac{1}{6}\ , \text{ and}$$

$$d_2(f_n, 0)^2 = \int_0^1 f_n(x)^2\,dx = \int_0^{1/n} n^4x^2 - 2n^5x^3 + n^6x^4\;dx = \Big[\tfrac{n^4}{3}x^3 - \tfrac{n^5}{2}x^4 + \tfrac{n^6}{5}x^5\Big]_0^{1/n} = \tfrac{n}{30} \to \infty.$$

4: (a) For each $n\in\mathbb{Z}^+$, let $x_n=(x_{n,k})_{k\geq 1}\in\mathbb{R}^\infty$ be given by $x_n=\sum\limits_{k=1}^{n}\frac{k+1}{k}\,e_k$, where e_k is the k^{th} standard basis vector in $\mathbb{R}^\infty$ $\big($so we have $x_{n,k}=\frac{k+1}{k}$ when $k\leq n$ and $x_{n,k}=0$ when $k>n\big)$. Find $\lim\limits_{n\to\infty}\big(\lim\limits_{k\to\infty}x_{n,k}\big)$ in $\mathbb{R}$, and find $\lim\limits_{k\to\infty}\big(\lim\limits_{n\to\infty}x_{n,k}\big)$ in $\mathbb{R}$, and determine whether the sequence $(x_n)_{n\geq 1}$ converges in (ℓ_∞,d_∞).

Solution: Given $n\in\mathbb{Z}^+$, since $x_{n,k}=0$ for all $k>n$, we have $\lim\limits_{k\to\infty}x_{n,k}=0$, and so $\lim\limits_{n\to\infty}\big(\lim\limits_{k\to\infty}x_{n,k}\big)=0$. Given $k\in\mathbb{Z}^+$, since $x_{n,k}=\frac{k+1}{k}$ for all $n\geq k$, we have $\lim\limits_{n\to\infty}x_{n,k}=\frac{k+1}{k}$, so $\lim\limits_{k\to\infty}\big(\lim\limits_{n\to\infty}x_{n,k}\big)=\lim\limits_{k\to\infty}\frac{k+1}{k}=1$. We claim that $(x_n)_{n\geq 1}$ does not converge in (ℓ_∞,d_∞). Suppose, for a contradiction, that $x_n\to a$ in (ℓ_∞,d_∞). By Theorem 5.6, for all $k\in\mathbb{Z}^+$ we must have $a_k=\lim\limits_{n\to\infty}x_{n,k}=\frac{k+1}{k}$, and so $a=(a_k)_{k\geq 1}=\big(\frac{2}{1},\frac{3}{2},\frac{4}{3},\frac{5}{4},\cdots\big)$. For all $n\in\mathbb{Z}^+$ since $x_{n,k}=a_k=\frac{k+1}{k}$ for $k\leq n$ and $x_{n,k}=0$ for $k>n$, we have $|x_{n,k}-a_k|=0$ for $k\leq n$ and $|x_{n,k}-a_k|=\frac{k+1}{k}$ for $k>n$, and so $\|x_n-a\|_\infty=\sup\big\{\frac{k+1}{k}\big|k\geq n+1\big\}=\frac{n+2}{n+1}>1$. Since $\|x_n-a\|_\infty>1$ for all $n\in\mathbb{Z}^+$, it follows that $x_n\not\to a$ in (ℓ_∞,d_∞), so we have obtained the desired contradiction.

(b) Let $K=\big\{x=(x_k)_{k\geq 1}\in\ell_\infty\big|\lim\limits_{k\to\infty}x_k=0\big\}$. Show that in (ℓ_∞,d_∞) we have $\overline{\mathbb{R}^\infty}=K$.

Solution: Let $a=(a_k)_{k\geq 1}\in K$. For each $n\in\mathbb{Z}^+$, let $x_n=\sum_{k=1}^n a_ke_k=\big(a_1,a_2,\cdots,a_n,0,0,\cdots\big)$ and note that $(x_n)_{n\geq 1}$ is a sequence in $\mathbb{R}^\infty$. We claim that $x_n\to a$ in ℓ_∞ (using the metric d_∞). Let $\epsilon>0$. Since $a\in K$ so that $\lim_{k\to\infty}a_k=0$, we can choose $m\in\mathbb{Z}^+$ so that for all $k\in\mathbb{Z}^+$, $k\geq m\Longrightarrow|a_k|<\frac{\epsilon}{2}$. Then for $n\in\mathbb{Z}^+$ with $n\geq m$ we have $\|x_n-a\|_\infty=\|a-x_n\|_\infty=\big\|(0,0,\cdots,0,a_{n+1},a_{n+2},\cdots)\big\|_\infty=\sup\big\{|a_k|\,\big|\,k>m\big\}\leq\frac{\epsilon}{2}<\epsilon$. Thus $x_n\to a$ in ℓ_∞ (using d_∞), as claimed. Since $(x_n)_{n\geq 1}$ is a sequence in $\mathbb{R}^\infty$ with $x_n\to a$ in ℓ_∞, we have $a\in\overline{\mathbb{R}^\infty}$ (by Part 2 of Theorem 5.16). Since $a\in K$ was arbitrary, we have $K\subset\overline{\mathbb{R}^\infty}$.

Now let $a=(a_k)_{k\geq 1}\in\overline{\mathbb{R}^\infty}$. We claim that $a_k\to 0$ in $\mathbb{R}$ so that $a\in K$. Let $\epsilon>0$. Since $a\in\overline{\mathbb{R}^\infty}$, by Part 2 of Theorm 4.47, we have $B_\infty(a,\epsilon)\cap\mathbb{R}^\infty\neq\emptyset$, so we can choose $b=(b_k)_{k\geq 1}\in\mathbb{R}^\infty$ with $\|a-b\|_\infty<\epsilon$. Since $\|a-b\|_\infty<\epsilon$, we have $|a_k-b_k|\leq\|a-b\|_\infty<\epsilon$ for all $k\in\mathbb{Z}^+$. Since $b\in\mathbb{R}^\infty$, we can choose $m\in\mathbb{Z}^+$ so that for all $k\in\mathbb{Z}^+$ we have $k\geq m\Longrightarrow b_k=0$. Then for $k\geq m$ we have $|a_k|=|a_k-b_k|<\epsilon$. Thus $a_k\to 0$ in $\mathbb{R}$, so that $a\in K$, as claimed. Since $a\in\overline{\mathbb{R}^\infty}$ was arbitrary, we have $\overline{\mathbb{R}^\infty}\subseteq K$.