AMATH/PMATH 331 Assignment 4 — Due Nov 18, 11:00 pm

**1:** (a) Let $A=\left\{(x,y)\in\mathbb{R}^2\,\middle|\,4x^2+y^2<8x\right\}$. Prove that $A$ is open in $\mathbb{R}^2$.

(b) Let $B=\left\{(a,b,c,d)\in\mathbb{R}^4\,\middle|\,\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)^2=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right\}$. Prove that $B$ is closed in $\mathbb{R}^4$.

(c) Let $C=\left\{(t^2-1\,,\,t^3-t)\in\mathbb{R}^2\,\middle|\,t\in\mathbb{R}\right\}$. Determine whether $C$ is closed in $\mathbb{R}^2$.

**2:** We consider that $\mathbb{C}=\mathbb{R}^2$ (when $x,y\in\mathbb{R}$, the ordered pair $(x,y)\in\mathbb{R}$ is equal to the complex number $z=x+iy\in\mathbb{C}$ and we have $\overline{z}=x-iy\in\mathbb{C}$), and the usual norm in $\mathbb{C}$ is equal to the usual norm in $\mathbb{R}^2$: for $z=x+iy=(x,y)$ we have $\|z\|=\sqrt{x^2+y^2}$. Recall that for $z,w\in\mathbb{C}$ we have $\|z\|^2=z\,\overline{z}$ and $\|zw\|=\|z\|\,\|w\|$.

(a) For $n\geq 1$, let $s_n=\sum\limits_{k=1}^{n}\left(\frac{1+i}{3}\right)^k$. Prove, from the definition of a limit, that $\lim\limits_{n\to\infty}s_n=\frac{1+3\,i}{5}$ in $\mathbb{C}$.

(b) Define $f:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ by $f(z)=\frac{z^2-\overline{z}^2}{\|z\|^2}$. Prove, from the definition of a limit, that $\lim\limits_{z\to 0}f(z)$ does not exist.

**3:** (a) Define $f_n:[0,1]\subseteq\mathbb{R}\to\mathbb{R}$ by $f_n(x)=1-nx$ for $0\leq x\leq\frac{1}{n}$ and $f_n(x)=0$ for $\frac{1}{n}\leq x\leq 1$. Show that $f_n\to 0$ in $\mathcal{C}[0,1]$ using either of the metrics $d_1$ or $d_2$, but $f_n\not\to 0$ pointwise on $[0,1]$.

(b) Define $f_n:[0,1]\subseteq\mathbb{R}\to\mathbb{R}$ by $f_n(x)=n^2x-n^3x^2$ for $0\leq x\leq\frac{1}{n}$ and $f_n(x)=0$ for $\frac{1}{n}\leq x\leq 1$. Show that $f_n\to 0$ pointwise on $[0,1]$ but $f_n\not\to 0$ in $\mathcal{C}[0,1]$ using either of the metrics $d_1$ or $d_2$.

**4:** (a) For each $n\in\mathbb{Z}^+$, let $x_n=(x_{n,k})_{k\geq 1}\in\mathbb{R}^\infty$ be given by $x_n=\sum\limits_{k=1}^{n}\frac{k+1}{k}\,e_k$, where $e_k$ is the $k^{\text{th}}$ standard basis vector in $\mathbb{R}^\infty$ (so we have $x_{n,k}=\frac{k+1}{k}$ when $k\leq n$ and $x_{n,k}=0$ when $k>n$). Find $\lim\limits_{n\to\infty}\left(\lim\limits_{k\to\infty}x_{n,k}\right)$ in $\mathbb{R}$, and find $\lim\limits_{k\to\infty}\left(\lim\limits_{n\to\infty}x_{n,k}\right)$ in $\mathbb{R}$, and determine whether the sequence $(x_n)_{n\geq 1}$ converges in $(\ell_\infty,d_\infty)$.

(b) Let $K=\left\{x=(x_k)_{k\geq 1}\in\ell_\infty\,\middle|\,\lim\limits_{k\to\infty}x_k=0\right\}$. Show that in $(\ell_\infty,d_\infty)$ we have $\overline{\mathbb{R}^\infty}=K$.