AMATH/PMATH 331 Solutions to Assignment 3

1: (a) Let $M_{k\times\ell}(\mathbb{R})$ be the vector space of real $k\times\ell$ matrices. For $A,B\in M_{k\times\ell}(\mathbb{R})$, define $d(A,B)=\text{rank}(A-B)$. Show that d is a metric on $M_{k\times\ell}(\mathbb{R})$.

Solution: It is clear that d is positive definite, that is $d(A,B)\geq 0$ for all $A,B\in M_{k\times\ell}(\mathbb{R})$ with $d(A,B)=0$ if and only if $A=B$, because only the zero matrix has rank zero. It is also clear that d is symmetric, that is $d(A,B)=d(B,A)$, since for any matrix X we have $\text{rank}(X)=\text{rank}(-X)$. We need to verify that d satisfies the triangle inequality. Let $A,B,C\in M_{k\times\ell}(\mathbb{R})$. Let $X=A-B$, $Y=B-C$ and $Z=C-A$. Note that $X+Y=Z$. Let $u_1,\cdots,u_\ell$ be the columns of X, let $v_1,\cdots,v_\ell$ be the columns of Y and let $w_1,\cdots,w_\ell$ be the columns of Z. Since $X+Y=Z$ we have $w_i=u_i+v_i$ for all indices i, and so $\text{Span}\{w_1,\cdots,w_\ell\}\subseteq\text{Span}\{u_1,\cdots,u_\ell,v_1,\cdots,v_\ell\}$. Let $U=\text{Col}(A)=\text{Span}\{u_1,\cdots,u_\ell\}$, $V=\text{Col}(B)=\text{Span}\{v_1,\cdots,v_\ell\}$ and $W=\text{Col}(Z)=\text{Span}\{w_1,\cdots,w_\ell\}$. Since $W=\text{Span}\{w_1,\cdots,w_\ell\}\subseteq\text{Span}\{u_1,\cdots,u_\ell,v_1,\cdots,v_\ell\}=U+V$ we have

$$\text{rank}(Z)=\dim W\leq\dim(U+V)=\dim U+\dim V-\dim(U\cap V)\leq\dim U+\dim V=\text{rank}(X)+\text{rank}(Y),$$

and so

$$d(A,C)=\text{rank}(A-C)=\text{rank}(Z)\leq\text{rank}(X)+\text{rank}(Y)=\text{rank}(A-B)+\text{rank}(B-C)=d(A,B)+d(B,C).$$

(b) Let d be a metric on a set X. For $x,y\in X$, let $d_0(x,y)=\min\big\{d(x,y),1\big\}$. Show that d_0 is a metric on X which induces the same topology as d.

Solution: It is clear that d_0 is symmetric and positive definite. Let us verify that d_0 satisfies the Triangle Inequality. Let $x,y,z\in X$. If $d(x,y)\geq 1$ then $d_0(x,y)=1$ hence $d_0(x,z)\leq 1\leq d_0(x,y)+d_0(y,z)$. If $d(y,z)\geq 1$ then $d_0(y,z)=1$ hence $d_0(x,z)\leq 1\leq d_0(x,y)+d_0(y,z)$. If $d(x,y)<1$ and $d(y,z)<1$ then $d_0(x,y)=d(x,y)$ and $d_0(y,z)=d(y,z)$ hence $d_0(x,z)\leq d(x,z)\leq d(x,y)+d(y,z)=d_0(x,y)+d_0(y,z)$. In all cases, we have $d_0(x,z)\leq d_0(x,y)+d_0(y,z)$, as required, hence d_0 is a metric on X.

We claim that d_0 induces the same topology as d. For $a\in X$ and $r>0$, let $B(a,r)=\{x\in X\,|\,d(x,a)<r\}$ and $B_0(a,r)=\{x\in X\,|\,d_0(x,a)<r\}$. Let $U\subseteq X$. Suppose that U is open in (X,d_0). Let $a\in U$. Choose $r>0$ so that $B_0(a,r)\subseteq U$. Let $x\in B(a,r)$. Then we have $d_0(x,a)\leq d(x,a)<r$ so that $x\in B_0(a,r)\subseteq U$. This shows that $B(a,r)\subseteq U$, hence U open in (X,d). Suppose, conversely, that U is open in (X,d). Let $a\in U$. Choose $s>0$ such that $B(a,s)\subseteq U$. Choose r with $0<r\leq 1$ and with $r\leq s$. Let $x\in B_0(a,r)$. Since $d_0(x,a)=\min\big\{d(x,a),1\big\}$ and $d_0(x,a)<r\leq 1$, we must have $d_0(x,a)=d(a,x)$, hence $d(a,x)=d_0(a,x)<r$ so that $x\in B(a,r)\subseteq B(a,s)\subseteq U$. This shows that $B_0(a,r)\subseteq U$, hence U is open in (X,d_0).

2: (a) Let $A = \{(x,y) \in \mathbb{R}^2 \,|\, x>0, y>0, xy < 1\}$. Prove, from the definition of an open set, that A is open in $\mathbb{R}^2$.

Solution: Let $(a,b) \in A$, so we have $a > 0$, $b > 0$ and $ab < 1$. Note that for $r > 0$, when $(x,y) \in B\big((a,b), r\big)$, we have

$$|x-a| = \sqrt{|x-a|^2} \le \sqrt{|x-a|^2 + |y-a|^2} = d\big((x,y),(a,b)\big) < r$$

so that $a - r < x < a + r$, and similarly $|y - b| < r$ so that $b - r < y < b + r$. Consider the case that $a \le b$. Choose $r = \min\left\{a, \frac{1-ab}{2a+b}\right\}$. Then when $(x,y) \in B\big((a,b), r\big)$, we have $x > a - r \ge a - a = 0$ and $y > b - r \ge b - b = 0$, and $xy < (a+r)(b+r) = ab + r(a+b+r) \le ab + r(2a+b) \le ab + (1-ab) = 1$, so that $(x,y) \in A$. Thus we have $B\big((a,b), r\big) \subseteq A$. Similarly, in the case that $b \le a$ we choose $r = \min\left\{b, \frac{1-ab}{a+2b}\right\}$ to get $B\big((a,b), r\big) \subseteq A$. Thus A is open.

(b) For each $k \in \{1, 2, \cdots, n\}$, let $a_k, b_k \in \mathbb{R}$ with $a_k \le b_k$, and let $I_k = [a_k, b_k] \subseteq \mathbb{R}$. Let A be the closed bounded rectangle $A = I_1 \times I_2 \times \cdots \times I_n = \left\{x \in \mathbb{R}^n \,\middle|\, \text{each } x_k \in I_k\right\}$. Prove, from the definition of open and closed sets, that A is closed in $\mathbb{R}^n$.

Solution: We need to show that the set $A^c = \mathbb{R}^n \setminus A$ is open in $\mathbb{R}^n$, so we must show that for all $p \in A^c$ there exists $r > 0$ such that $B(p,r) \subseteq A^c$. Let $p \in A^c$. Since $p \in A^c$, we can choose $\ell \in \{1, 2, \cdots, n\}$ such that $p_\ell \notin I_\ell = [a_\ell, b_\ell]$. Then either $p_\ell < a_\ell$ or $p_\ell > b_\ell$. Say $p_\ell < a_\ell$ (the case that $p_\ell > b_\ell$ is similar). Let $r = a_\ell - p_\ell$, and note that $r > 0$. Let $x \in B(p,r) = B_2(p,r)$ so that $\|x - p\|_2 < r$. Then we have $|x_\ell - p_\ell| = \sqrt{|x_\ell - p_\ell|^2} \le \sqrt{\sum_{k=1}^n |x_k - p_k|^2} = \|x - p\|_2 < r$ so that $x_\ell < p_\ell + r = a_\ell$. Since $x_\ell < a_\ell$ so that $x_\ell \notin I_\ell = [a_\ell, b_\ell]$, we have $x \in A^c$. Since $x \in B(p,r)$ was arbitrary, we have $B(p,r) \subseteq A^c$, as required.

3: (a) Let X be a metric space, let $A \subseteq X$ and let $a \in X$. Using the definition of open and closed sets, the definition of the closure $\overline{A}$, and the fact that open balls in X are open in X, prove that $a \in \overline{A}$ if and only if for every $r > 0$ we have $B(a,r) \cap A \neq \emptyset$.

Solution: Suppose that $a \in \overline{A}$. Let $r > 0$. Suppose, for a contradiction, that $B(a,r) \cap A = \emptyset$. Then we have $A \subseteq B(a,r)^c$ where $B(a,r)^c = X \setminus B(a,r)$. Note that (from the definition of a closed set) $B(a,r)^c$ is closed because $B(a,r)$ is open in X. Since $a \in \overline{A}$ and $B(a,r)^c$ is a closed set in X with $A \subseteq B(a,r)^c$, it follows (from the definition of $\overline{A}$) that $a \in B(a,r)^c$. This gives the desired contradiction, since $a \in B(a,r)$.

Suppose, conversely, that for every $r > 0$ we have $B(a,r) \cap A \neq \emptyset$. Let K be a closed set in X with $A \subseteq K$. Since K is closed in X it follows (from the definition of a closed set) that K^c is open in X. Suppose, for a contradiction, that $a \notin K$. Since $a \in K^c = X \setminus K$, which is open, it follows (from the definition of an open set) that we can choose $r > 0$ such that $B(a,r) \subseteq K^c$. Since $B(a,r) \subseteq K^c$, we have $B(a,r) \cap K = \emptyset$. Since $B(a,r) \cap K = \emptyset$ and $A \subseteq K$, we have $B(a,r) \cap A = \emptyset$. This contradicts our supposition that for all $r > 0$ we have $B(a,r) \cap A \neq \emptyset$. Thus $a \in K$. Since $a \in K$ for every closed set K in X with $A \subseteq K$, it follows (from the definition of $\overline{A}$) that $a \in \overline{A}$.

(b) Let $A = \left\{ \left(x, \sin \frac{1}{x}\right) \in \mathbb{R}^2 \,\middle|\, 0 < x \leq \frac{1}{\pi} \right\}$ and $B = \left\{ (0,y) \in \mathbb{R}^2 \,\middle|\, -1 \leq y \leq 1 \right\}$. Prove that $\overline{A} = A \cup B$ in $\mathbb{R}^2$.

Solution: We claim that $A \cup B \subseteq \overline{A}$. From the definition of $\overline{A}$ we have $A \subseteq \overline{A}$, so it suffices to show that $B \subseteq \overline{A}$. Let $b \in B$, say $b = (0,y)$ with $-1 \leq y \leq 1$. By Part (a), to show that $b \in \overline{A}$ it suffices to show that for every $r > 0$ we have $B(b,r) \cap A \neq \emptyset$. Let $r > 0$. Note that when $x = \frac{2}{(2n+1)\pi}$ with $n \in \mathbb{Z}^+$, we have $\sin \frac{1}{x} = (-1)^n$. Choose $n \in \mathbb{Z}^+$ so that $\frac{2}{(2n+1)\pi} < r$. Let $x_1 = \frac{2}{(2n+3)\pi}$ and $x_2 = \frac{2}{(2n+1)\pi}$. Then $0 < x_1 < x_2 < r$ and one of the two numbers $\sin \frac{1}{x_1}$ and $\sin \frac{1}{x_2}$ is equal to -1 and the other is equal to 1. By the Intermediate Value Theorem, since $\sin \frac{1}{x}$ is continuous for $x > 0$ and $-1 \leq y \leq 1$, we can choose x between x_1 and x_2 such that $\sin \frac{1}{x} = y$. Then for $a = (x,y) = \left(x, \sin \frac{1}{x}\right)$ we have $a \in A$ with $\|a - b\| = \|(x,y) - (0,y)\| = x < x_2 = \frac{2}{(2n+1)\pi} < r$ so that $a \in B(b,r) \cap A$. Thus $b \in \overline{A}$, as required.

We claim that $\overline{A} \subseteq A \cup B$. Let $a = (x,y) \in \overline{A}$. Let $R = [0, \frac{1}{\pi}] \times [-1,1] \subseteq \mathbb{R}^2$. Note that R is closed (by Question 2(b)) with $A \subseteq R$ so (from the definition of $\overline{A}$) we have $\overline{A} \subseteq R$. Since $a = (x,y) \in \overline{A} \subseteq R$ we have $0 \leq x \leq \frac{1}{\pi}$ and $-1 \leq y \leq 1$. If $x = 0$ then we have $(x,y) \in B$. It remains to show that if $0 < x \leq \frac{1}{\pi}$ then $(x,y) \in A$. Let $0 < x \leq \frac{1}{\pi}$ and $-1 \leq y \leq 1$ and suppose, for a contradiction, that $(x,y) \notin A$. Then $y \neq \sin \frac{1}{x}$. Let $\epsilon = \frac{1}{2}\left|y - \sin \frac{1}{x}\right|$ and note that $\epsilon > 0$. Since $\sin \frac{1}{t}$ is continuous for $t > 0$, we can choose $\delta > 0$ so that for all $t > 0$, if $|t - x| < \delta$ then $\left|\sin \frac{1}{t} - \sin \frac{1}{x}\right| < \epsilon = \frac{1}{2}\left|y - \sin \frac{1}{x}\right|$ so that, by the Triangle Inequality,

$$\left|y - \sin \tfrac{1}{t}\right| \geq \left|y - \sin \tfrac{1}{x}\right| - \left|\sin \tfrac{1}{x} - \sin \tfrac{1}{t}\right| > \tfrac{1}{2}\left|y - \sin \tfrac{1}{x}\right| = \epsilon.$$

Taking $r = \min\{\epsilon, \delta\}$, we have $B\big((x,y), r\big) \cap A = \emptyset$: indeed if we had $\left(t, \sin \frac{1}{t}\right) \in B\big((x,y), r\big)$ then we would have $|t - x| < \delta$ and $\left|\sin \frac{1}{t} - y\right| < \epsilon$, which is not possible. Since $B\big((x,y), r\big) \cap A = \emptyset$, it follows from Part (a) that $(x,y) \notin \overline{A}$, giving the desired contradiction.

4: (a) Show that in (ℓ_2, d_2) we have $\overline{\mathbb{R}^\infty} = \ell_2$ and $(\mathbb{R}^\infty)^\circ = \emptyset$.

Solution: We claim that in (ℓ_2, d_2) we have $\overline{\mathbb{R}^\infty} = \ell_2$. Let $a \in \ell_2$. We need to show that $a \in \overline{\mathbb{R}^\infty}$. It suffices to show that for all $r > 0$ we have $B_2(a, r) \cap \mathbb{R}^\infty \neq \emptyset$. Let $r > 0$. Since $a \in \ell_2$ so that $\sum {a_k}^2$ converges, we can choose $n \in \mathbb{Z}^+$ such that $\sum_{k=n+1}^{\infty} {a_k}^2 < r^2$. Let $x = \sum_{k=1}^{n} a_k e_k = (a_1, a_2, \cdots, a_n, 0, 0, \cdots)$. Then $x \in \mathbb{R}^\infty$ and $\|a - x\|_2 = \left(\sum_{k=n+1}^{\infty} {a_k}^2\right)^{1/2} < r$ so that $x \in B_2(a, r)$.

We claim that in (ℓ_2, d_2) we have $(\mathbb{R}^\infty)^\circ = \emptyset$. Let $a \in \mathbb{R}^\infty$. We need to show that for every $r > 0$, the ball $B_2(a, r)$ is not contained in $\mathbb{R}^\infty$, that is for every $r > 0$ there exists $x \in B_2(a, r)$ with $x \notin \mathbb{R}^\infty$. Let $r > 0$. Let $u = \sum_{k=1}^{\infty} \frac{1}{2^{k/2}} e_k = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2\sqrt{2}}, \cdots\right)$ and note that $u \notin \mathbb{R}^\infty$ and $\|u\|_2 = \left(\sum_{k=1}^{\infty} \frac{1}{2^k}\right)^{1/2} = 1$. Let $x = a + \frac{r}{2} u$. Then $x \notin \mathbb{R}^\infty$ (since if we had $x \in \mathbb{R}^\infty$ then we would also have $u = \frac{2}{r}(x - a) \in \mathbb{R}^\infty$) and we have $\|x - a\|_2 = \left\|\frac{r}{2}\, u\right\|_2 = \frac{r}{2}$ so that $x \in B_2(a, r)$.

(b) Let $A = \left\{x = (x_n)_{n \geq 1} \in \mathbb{R}^\omega \,\middle|\, \forall n \in \mathbb{Z}^+ \ |x_n| \leq \frac{1}{2^n}\right\}$. Show that in (ℓ_1, d_1) we have $A^\circ = \emptyset$ and $\overline{A} = A$.

Solution: We claim that $A^\circ = \emptyset$. Let $a = (a_n)_{n \geq 1} \in A$. We need to show that for every $r > 0$ the ball $B_1(a, r)$ is not contained in A, that is for every $r > 0$ there exists $x \in B_1(a, r)$ with $x \notin A$. Let $r > 0$. Choose $m \in \mathbb{Z}^+$ so that $\frac{1}{2^m} < \frac{r}{2}$. Define $x = (x_n)_{n \geq 1}$ by $x_n = a_n$ when $n \neq m$ and $x_m = \frac{r}{2}$. Then $x \in \ell_1$ since $\sum_{n=1}^{\infty} |x_n| = \sum_{n=1}^{\infty} |a_n| - |a_m| + \frac{r}{2} < \infty$, and $x \notin A$ since $x_m = \frac{r}{2} > \frac{1}{2^m}$, and $x \in B_1(a, r)$ since

$$\|x - a\|_1 = \sum_{n=1}^{\infty} |x_n - a_n| = \left|\tfrac{r}{2} - a_m\right| \leq \tfrac{r}{2} + |a_m| \leq \tfrac{r}{2} + \tfrac{1}{2^m} < \tfrac{r}{2} + \tfrac{r}{2} = r\,.$$

To show that $\overline{A} = A$, we must show that A is closed, or equivalently that A^c is open. Let $a = (a_n)_{n \geq 1} \in A^c$. It suffices to show that there exists $r > 0$ such that $B_1(a, r) \subseteq A^c$. Since $a \notin A$, we can choose $m \in \mathbb{Z}^+$ so that $|a_m| > \frac{1}{2^m}$. Let $r = |a_m| - \frac{1}{2^m}$ and note that $r > 0$. We claim that $B_1(a, r) \subseteq A^c$. Let $x = (x_n)_{n \geq 1} \in B_1(a, r)$. Then

$$|a_m| - |x_m| \leq |a_m - x_m| \leq \sum_{n=1}^{\infty} |a_n - x_n| = \|a - x\|_1 < r = |a_m| - \tfrac{1}{2^m}\,.$$

It follows that $|x_m| > \frac{1}{2^m}$, and so $x = (x_n)_{n \geq 1} \notin A$, that is $x \in A^c$ as required.