AMATH/PMATH 331 Assignment 3 Due Nov 4, 11:00 pm

1: (a) Let $M_{k\times\ell}(\mathbb{R})$ be the vector space of real $k\times\ell$ matrices. For $A,B\in M_{k\times\ell}(\mathbb{R})$, define $d(A,B)=\operatorname{rank}(A-B)$. Show that d is a metric on $M_{k\times\ell}(\mathbb{R})$.

(b) Let d be a metric on a set X. For $x,y\in X$, let $d_0(x,y)=\min\{d(x,y),1\}$. Show that d_0 is a metric on X which induces the same topology as d.

2: (a) Let $A=\{(x,y)\in\mathbb{R}^2 \,|\, x>0, y>0, xy<1\}$. Prove, from the definition of an open set, that A is open in $\mathbb{R}^2$.

(b) For each $k\in\{1,2,\cdots,n\}$, let $a_k,b_k\in\mathbb{R}$ with $a_k\le b_k$, and let $I_k=[a_k,b_k]\subseteq\mathbb{R}$. Let A be the closed bounded rectangle $A=I_1\times I_2\times\cdots\times I_n=\{x\in\mathbb{R}^n \,|\, \text{each } x_k\in I_k\}$. Prove, from the definition of open and closed sets, that A is closed in $\mathbb{R}^n$.

3: (a) Let X be a metric space, let $A\subseteq X$ and let $a\in X$. Using the definition of open and closed sets, the definition of the closure $\overline{A}$, and the fact that open balls in X are open in X, prove that $a\in\overline{A}$ if and only if for every $r>0$ we have $B(a,r)\cap A\ne\emptyset$.

(b) Let $A=\{(x,\sin\frac{1}{x})\in\mathbb{R}^2 \,|\, 0<x\le\frac{1}{\pi}\}$ and $B=\{(0,y)\in\mathbb{R}^2 \,|\, -1\le y\le 1\}$. Prove that $\overline{A}=A\cup B$ in $\mathbb{R}^2$.

4: (a) Show that in (ℓ_2,d_2) we have $\overline{\mathbb{R}^\infty}=\ell_2$ and $(\mathbb{R}^\infty)^o=\emptyset$.

(b) Let $A=\{x=(x_n)_{n\ge1}\in\mathbb{R}^\omega \,|\, \forall n\in\mathbb{Z}^+\ |x_n|\le\frac{1}{2^n}\}$. Show that in (ℓ_1,d_1) we have $A^\circ=\emptyset$ and $\overline{A}=A$.