1: (a) Let $A \subseteq \mathbb{R}$ be an open interval, let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, let $M \geq 0$, and suppose that f is differentiable in A with $|f'(x)| \leq M$ for all $x \in A$. Prove that f is iniformly continuous on A. (Hint: use the Mean Value Theorem).

(b) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \cos\left(\pi x^2\right)$. Prove that f is not uniformly continuous on $\mathbb{R}$.

(c) Let $f : [a, b) \to \mathbb{R}$ be uniformly continuous. Prove that f is bounded.

2: (a) Define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = nxe^{-nx}$. Find the pointwise limit $f(x) = \lim\limits_{n\to\infty} f_n(x)$ and determine whether $f_n \to f$ uniformly on $[0, \infty)$.

(b) Define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = \frac{x}{1+nx^2}$. Find the pointwise limit $f(x) = \lim\limits_{n\to\infty} f_n(x)$ and determine whether $f_n \to f$ uniformly on $[0, \infty)$.

(c) Define $f_n : [0, \infty] \to \mathbb{R}$ by $f_n(x) = \frac{x+n}{x+4n}$. Show that (f_n) converges uniformly on $[0, r]$ for every $r > 0$ but that (f_n) does not converge uniformly on $[0, \infty)$.

3: (a) Suppose that $f_n \to f$ uniformly on $A \subseteq \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$. Suppose that f is bounded and g is continuous. Prove that $g \circ f_n \to g \circ f$ uniformly on A.

(b) (The Riemann Zeta Function) Define $\zeta : (1, \infty) \to \mathbb{R}$ by $\zeta(x) = \sum\limits_{n=1}^{\infty} \frac{1}{n^x}$. Prove that ζ is differentiable on $(1, \infty)$. Hint: use the Weierstrass M-Test, together with convergence tests from first year calculus, to show that for all $r > 1$ the series $\sum \frac{1}{n^x}$ and $\sum \frac{-\ln n}{n^x}$ both converge uniformly on $[r, \infty)$, then apply Theorem 3.16 (Uniform Convergence and Differentiation).