AMATH/PMATH 331 Solutions to Assignment 2

1: Let $a \in \mathbb{R}$. For each of the following statements, provide a simple description of the functions $f : \mathbb{R} \to \mathbb{R}$ for which the statement is true.

(a) $\forall \epsilon > 0\ \forall \delta > 0\ \forall x \in \mathbb{R}\ \big(|x-a| \leq \delta \Longrightarrow \big|f(x) - f(a)\big| \leq \epsilon\big)$

Solution: We claim that this statement is true if and only if f is constant. Suppose that f is constant. Let $\epsilon > 0$, let $\delta > 0$ and let $x \in \mathbb{R}$. Suppose that $|x - a| \leq \delta$. Since f is constant, we have $f(x) = f(a)$ and so $|f(x) - f(a)| = 0 \leq \epsilon$.

Conversely, suppose that f is not constant. Since f is not constant, we can choose $x \in \mathbb{R}$ such that $f(x) \neq f(a)$. Since $f(x) \neq f(a)$ we also have $x \neq a$. Choose $\epsilon = \frac{|f(x)-f(a)|}{2}$ and choose $\delta = |x - a|$. Then $|x - a| \leq \delta$ and $|f(x) - f(a)| > \epsilon$. We have shown that $\exists \epsilon{>}0\ \exists \delta{>}0\ \exists x{\in}\mathbb{R}\big(|x - a| \leq \delta \text{ and } |f(x) - f(a)| > \epsilon\big)$, in other words, we have shown that the given statement is false.

(b) $\exists \epsilon > 0\ \forall \delta > 0\ \forall x \in \mathbb{R}\ \big(|x-a| \leq \delta \Longrightarrow \big|f(x) - f(a)\big| \leq \epsilon\big)$

Solution: A function $f : \mathbb{R} \to \mathbb{R}$ is *bounded* when there exists $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. We claim that the given statement is true if and only if f is bounded. Suppose that f is bounded. Choose $M \in \mathbb{R}$ so that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Choose $\epsilon = 2M$. Let $\delta > 0$ and let $x \in \mathbb{R}$. Suppose that $|x - a| \leq \delta$. Since $|f(x)| \leq M$ and $|f(a)| \leq M$ we have $|f(x) - f(a)| \leq |f(x)| + |f(a)| \leq 2M = \epsilon$.

Conversely, suppose that f is not bounded. This means that for all $M \geq 0$ there exists $x \in \mathbb{R}$ with $|f(x)| > M$. Let $\epsilon > 0$. Since f is not bounded we can choose $x \in \mathbb{R}$ so that $|f(x)| > |f(a)| + \epsilon$. Note that $|f(x)| = |f(x) - f(a) + f(a)| \leq |f(x) - f(a)| + |f(a)|$ and so we have $|f(x) - f(a)| \geq |f(x)| - |f(a)| > \epsilon$. Note that since $|f(x)| \neq |f(a)|$ we must have $x \neq a$, so we can choose $\delta = |x - a| > 0$ to get $|x - a| \leq \delta$. We have shown that $\forall \epsilon{>}0\ \exists \delta{>}0\ \forall x{\in}\mathbb{R}\ \big(|x - a| \leq \delta \text{ and } |f(x) - f(a)| > \epsilon\big)$, in other words, we have shown that the given statement is false.

(c) $\forall a \in \mathbb{R}\ \forall x \in \mathbb{R}\ \exists \epsilon > 0\ \forall \delta > 0\ \big(|f(x) - f(a)| \leq \epsilon \Longrightarrow |x - a| \leq \delta\big)$.

Solution: We claim that the given statement is true if and only if f is injective. Suppose that f is injective. Let $a \in \mathbb{R}$ and let $x \in \mathbb{R}$. If $x = a$ then we choose $\epsilon = 1$, and then for all $\delta > 0$ we have $|x - a| = 0 \leq \delta$ and so the statement "$|f(x) - f(a)| \Longrightarrow |x - a| \leq \delta$" is true. If $x \neq a$ then, since f is injective, we have $f(x) \neq f(a)$ so we can choose $\epsilon = \frac{|f(x)-f(a)|}{2}$ and then for all $\delta > 0$ we have $|f(x) - f(a)| > \epsilon$ and so once again the statement "$|f(x) - f(a)| \Longrightarrow |x - a| \leq \delta$" is true.

Conversely, suppose that f is not injective. Choose $a \in \mathbb{R}$ and $x \in \mathbb{R}$ so that $x \neq a$ and $f(x) = f(a)$. Let $\epsilon > 0$, and choose $\delta = \frac{|x-a|}{2}$. Then we have $|f(x) - f(a)| = 0 \leq \epsilon$ and $|x - a| > \delta$. We have shown that $\exists a{\in}\mathbb{R}\ \exists x{\in}\mathbb{R}\ \forall \epsilon{>}0 \exists \delta{>}0\ \big(|f(x) - f(a)| \leq \epsilon \text{ and } |x - a| > \delta\big)$, in other words, we have shown that the given statement is false.

2: For a sequence $(x_n)_{n\geq p}$ in $\mathbb{R}$ and for $a\in\mathbb{R}$, we say that a is a **subsequential limit point** of $(x_n)_{n\geq p}$ when

$$\forall\,\epsilon{>}0\ \forall\,m{\in}\mathbb{Z}\ \exists\,n{\in}\mathbb{Z}_{\geq p}\ (n{\geq}m \text{ and } |x_n-a|{\leq}\epsilon).$$

We denote the set of subsequential limit points of $(x_k)_{n\geq p}$ by $\text{Slim}\big((x_n)_{n\geq p}\big)$.

(a) Determine whether, for every sequence $(x_n)_{n\geq p}$ in $\mathbb{R}$, we have

$$\lim_{n\to\infty} x_n = a \implies \text{Slim}\big((x_n)_{n\geq p}\big)=\{a\}.$$

Solution: This is TRUE. Let $(x_n)_{n\geq p}$ be a sequence in $\mathbb{R}$ with $x_n\to a$. We claim that $\text{Slim}\big((x_n)_{n\geq p}\big)=\{a\}$. First we show that $\{a\}\subseteq\text{Slim}\big((x_n)_{n\geq p}\big)$. Let $\epsilon>0$ and let $m\in\mathbb{Z}_{\geq p}$. Since $x_n\to a$ we can choose $m_0\in\mathbb{Z}_{\geq p}$ so that $n\geq m_0\implies |x_n-a|<\epsilon$. Let $n=\max\{p,m,m_0\}$. Then $n\in\mathbb{Z}_{\geq p}$ with $n\geq m$ and $|x_n-a|\leq\epsilon$. This proves that $a\in\text{Slim}\big((x_n)_{n\geq p}\big)$, so we have $\{a\}\subseteq\text{Slim}\big((x_n)_{n\geq p}\big)$.

Conversely, we must show that $\text{Slim}\big((x_n)_{n\geq p}\big)\subseteq\{a\}$. Let $b\in\text{Sim}\big((x_n)_{n\geq p}\big)$. Suppose, for a contradiction, that $b\neq a$. Since $x_n\to a$, we can choose $m\in\mathbb{Z}_{n\geq p}$ so that for all $n\in\mathbb{Z}_{\geq p}$ we have $n\geq m\implies |x_n-a|<\frac{|b-a|}{2}$. Since $b\in\text{Slim}\big((x_n)_{n\geq p}\big)$, we can choose $n\in\mathbb{Z}_{\geq p}$ with $n\geq m$ and $|x_n-b|\leq\frac{|b-a|}{2}$. Then we have

$$|b-a|=|b-x_n+x_n-a|\leq|b-x_n|+|x_n-a|<\tfrac{|b-a|}{2}+\tfrac{|b-a|}{2}=|b-a|,$$

which is not possible. Thus we must have $b=a$, and this shows that $\text{Slim}\big((x_n)_{n\geq p}\big)\subseteq\{a\}$, as required.

(b) Determine whether, for every sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}$, we have

$$\text{Slim}\big((x_n)_{n\geq 1}\big)=\{a\}\implies\lim_{n\to\infty}x_n=a.$$

Solution: This statement is FALSE. For example, for the sequence $(x_n)_{n\geq 1}$ given by $x_n=a$ when n is even and $x_n=n$ when n is odd, $\text{Slim}\big((x_n)_{n\geq 1}\big)=\{a\}$ but $\lim\limits_{n\to\infty}x_n\neq a$, indeed the sequence $(x_n)_{n\geq 1}$ diverges.

Here is a proof that $\text{Slim}\big((x_n)_{n\geq 1}\big)=\{a\}$. Given $\epsilon>0$ and given $m\in\mathbb{Z}^+$ we can choose an even number $n\geq\max\{1,m\}$ to get $n\in\mathbb{Z}^+$ with $n\geq m$ and $|x_n-a|=|a-a|=0<\epsilon$. This shows that $a\in\text{Slim}\big((x_n)_{n\geq 1}\big)$ so that $\{a\}\subseteq\text{Slim}\big((x_n)_{n\geq 1}\big)$. Conversely, let $b\in\text{Slim}\big((x_n)_{n\geq 1}\big)$. Suppose, for a contradiction, that $b\neq a$. Let $\epsilon=\frac{|b-a|}{2}$ and let $m=|b-a|+|b|$. Then for $n\geq m$, if n is even then $|x_n-b|=|a-b|>\epsilon$, and if n is odd then $|x_n-b|=|n-b|\geq n-|b|\geq m-|b|=|b-a|+|b|-|b|=|b-a|>\epsilon$. But this contradicts the fact that $b\in\text{Slim}\big((x_n)_{n\geq 1}\big)$. Thus we must have $b=a$, and this shows that $\text{Slim}\big((x_n)_{n\geq 1}\big)\subseteq\{a\}$.

Here is a proof that $\lim_{n\to\infty}x_n\neq a$. Suppose, for a contradiction, that $x_n\to a$. Choose $m\in\mathbb{Z}^+$ so that $n\geq m\implies|x_n-a|\leq 1$. Then for all $n\geq m$ we have $a-1\leq x_n\leq a+1$. But we can choose an odd number $n\in\mathbb{N}$ with $n\geq\max\{m,a+2\}$ to get $n\geq m$ with $x_n=n>a+1$, giving the desired contradiction.

(c) Determine whether there exists a sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}$ with $\text{Slim}\big((x_n)_{n\geq 1}\big)=[0,1]\subseteq\mathbb{R}$.

Solution: This is TRUE. Let $(x_n)_{n\geq 1}$ be the sequence

$$(x_n)_{n\geq 1}=(x_1,x_2,x_3,\cdots)=\big(\tfrac01,\tfrac11,\tfrac02,\tfrac12,\tfrac22,\tfrac03,\tfrac13,\tfrac23,\tfrac33,\tfrac04,\tfrac14,\tfrac24,\tfrac34,\tfrac44,\tfrac05,\cdots\big).$$

Note that for each $q\in[0,1]\cap\mathbb{Q}$ there exist infinitely many indices $n\in\mathbb{Z}^+$ for which $x_n=q$: indeed for $\ell\in\mathbb{Z}^+$ and $k\in\{0,1,\cdots,\ell\}$, we have $x_{\ell(\ell+1)/2}=\frac{0}{\ell}$ and $x_{\ell(\ell+1)/2+k}=\frac{k}{\ell}$, and so when $q=\frac{k}{\ell}$, for $n=t\ell(t\ell+1)/2+tk$ with $t\in\mathbb{Z}^+$ we have $x_n=\frac{tk}{t\ell}=q$). We claim that for this sequence $(x_n)_{n\geq 1}$, we have $\text{Slim}\big((x_n)_{n\geq 1}\big)=[0,1]$.

Let $a\in[0,1]$. Let $\epsilon>0$ and let $m\in\mathbb{Z}^+$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose $q\in\mathbb{Q}$ with $q\in[0,1]$ and $|q-a|<\epsilon$. Since there are infinitely many indices $n\in\mathbb{Z}^+$ for which $x_n=q$, we can choose $n\in\mathbb{Z}^+$ with $n\geq m$ such that $x_n=q$, and then $|x_n-a|=|q-a|<\epsilon$. This proves that $[0,1]\subseteq\text{Slim}\big((x_n)_{n\geq 1}\big)$.

Let $a\notin[0,1]$. Then either $a<0$ or $a>1$. In the case that $a<0$ we choose $\epsilon=\frac{|a|}{2}$, and then for every $n\in\mathbb{Z}^+$ we have $x_n\geq 0$ so that $|x_n-a|=x_n-a\geq 0-a=|a|>\epsilon$. In the case that $a>1$ we choose $\epsilon=\frac{a-1}{2}$ and then for every $n\in\mathbb{Z}^+$ we have $x_n\leq 1$ so that $|x_n-a|=a-x_n\geq a-1>\epsilon$. In either case, we have $|x_n-a|>\epsilon$ for every $n\in\mathbb{Z}^+$ so that $a\notin\text{Slim}\big((x_n)_{n\geq 1}\big)$. This proves that $\text{Slim}\big((x_n)_{n\geq 1}\big)\subseteq[0,1]$.

3: For a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$, we say that f is **increasing** when $f(x) \le f(y)$ for all $x, y \in A$ with $x \le y$, we say that f is **strictly increasing** when $f(x) < f(y)$ for all $x, y \in A$ with $x < y$, we say that f is **decreasing** when $f(x) \ge f(y)$ for all $x, y \in A$ with $x \le y$, and we say that f is **strictly decreasing** when $f(x) > f(y)$ for all $x, y \in A$ with $x < y$. Let $a, b, u, v \in \mathbb{R}$ with $a < b$ and $u < v$, and let $f : (a, b) \subseteq \mathbb{R} \to (u, v) \subseteq \mathbb{R}$.

(a) Suppose that f is increasing, and let $(x_n)_{n\ge1}$ be a sequence in (a, b) with $x_n \to b$. Show that $\big(f(x_n)\big)_{n\ge1}$ converges in $\mathbb{R}$.

Solution: Note that $f\big((a, b)\big)$ is nonempty (for any $x \in (a, b)$ we have $f(x) \in f\big((a, b)\big)$) and $f\big((a, b)\big)$ is bounded (since $f\big((a, b)\big) \subseteq (u, v)$), and so $f\big((a, b)\big)$ has a supremum in $\mathbb{R}$. Let $w = \sup f\big((a, b)\big) = \sup\big\{f(x)\,\big|\,x \in (a, b)\big\}$. We claim that $f(x_n) \to w$. Let $\epsilon > 0$. By the Approximation Property of the supremum, we can choose $q \in f\big((a, b)\big)$ with $w - \epsilon < q \le w$. Since $q \in f\big((a, b)\big)$ we can choose $p \in (a, b)$ with $f(p) = w$. Since $x_n \to b$ and $p < b$ we can choose $m \in \mathbb{Z}^+$ so that $n \ge m \Longrightarrow |x_n - b| < b - p \Longrightarrow x_n > p$. Let $n \in \mathbb{Z}^+$ with $n \ge m$. Since f is increasing and $x_n > p$, we have $f(x_n) \ge f(p) = q > w - \epsilon$, Since $f(x_n) \in f\big((a, b)\big)$ and $w = \sup f\big((a, b)\big)$, we also have $f(x_n) \le w$, so that $w - \epsilon < f(x_n) \le w$. Thus for all $n \in \mathbb{Z}^+$ with $n \ge m$ we have $|f(x_n) - w| < \epsilon$, and so $f(x_n) \to w$, as claimed.

(b) Suppose that f is continuous and injective. Prove that either f is strictly increasing or f is strictly decreasing.

Solution: Suppose that f is not strictly decreasing. Choose $r, s \in (a, b)$ with $r < s$ such that $f(r) \le f(s)$. Note that since $r \ne s$ and f is injective, we have $f(r) \ne f(s)$, so that $f(r) < f(s)$. We must prove that f is strictly increasing.

We claim that for all $x, y, z \in (a, b)$ with $x < y < z$, either we have $f(x) < f(y) < f(z)$ or we have $f(x) > f(y) > f(z)$. Let $x, y, z \in (a, b)$ with $x < y < z$. Note that since x, y and z are distinct and f is injective, it follows that $f(x)$, $f(y)$ and $f(z)$ are distinct. Suppose that $f(x) < f(z)$ (the case that $f(x) > f(z)$ is similar). Suppose for a contradiction, that $f(y) < f(x)$. Choose p with $f(y) < p < f(x) < f(z)$. By the IVT (the Intermediate Value Theorem), since f is continuous, we can choose $t_1 \in (x, y)$ so that $f(t_1) = p$ and we can choose $t_2 \in (y, z)$ with $f(t_2) = p$. But then we have $t_1 \ne t_2$ with $f(t_1) = p = f(t_2)$ which contradicts the fact that f is injective. Thus we do not have $f(y) < f(x)$. A similar argument shows that we do not have $f(y) > f(z)$ and so (since $f(x)$, $f(y)$ and $f(z)$ are distinct) it follows that $f(x) < f(y) < f(z)$, as required. This proves the claim.

By the claim proven above, for $x \in (a, b)$, since $f(r) < f(s)$, if $x < r < s$ then $f(x) < f(r) < f(s)$, if $r < x < s$ then $f(r) < f(x) < f(s)$, and if $r < s < x$ then $f(r) < f(s) < f(s)$. It follows that when $x, y \in (a, b)$, if either $x \in \{r, s\}$ or $y \in \{r, s\}$ then we have $f(x) < f(y)$.

Let $x, y \in (a, b)$ with $x < y$ and $x, y \notin \{r, s\}$. When $x < r$ we can have $x < y < r$ (1), $x < r < y < s$ (2), or $x < r < s < y$ (3), and when $r < x < s$ we can have $r < x < y < s$ (4) or $r < x < s < y$ (5), and when $s < x$ we must have $s < x < y$ (6). We consider all 6 cases. Case 1: when $x < y < r$, since $f(x) < f(r)$ we must have $f(x) < f(y) < f(r)$. Case 2: when $x < r < y < s$, we have $f(x) < f(r) < f(y)$. Case 3: when $x < r < s < y$, we have $f(x) < f(r) < f(s) < f(y)$. Case 4: when $r < x < y < s$, since $f(r) < f(y)$ we must have $f(r) < f(x) < f(y)$. Case 5: when $r < x < s < y$, we have $f(r) < f(x) < f(s) < f(y)$. Case 6: when $s < x < y$, since $f(s) < f(y)$ we must have $f(s) < f(x) < f(y)$. In all cases we find that $f(x) < f(y)$, and hence f is strictly increasing.

(c) Suppose that f is increasing and surjective. Prove that f is continuous.

Solution: Let $x \in (a, b)$, and let $y = f(x)$. We need to show that f is continuous at x. Let $\epsilon > 0$. Choose $y_1, y_2 \in (u, v)$ such that $f(x) - \epsilon < y_1 < f(x) < y_2 < f(x) + \epsilon$. Since f is surjective, we can choose $x_1, x_2 \in (a, b)$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since f is increasing and $f(x_1) = y_1 < f(x) < y_2 = f(x_2)$, we must have $x_1 < x < x_2$. Let $\delta = \min\{x - x_1, x_2 - x\}$. Then for all $t \in (a, b)$ we have

$$\begin{aligned}
|t - x| < \delta &\Longrightarrow x - \delta < t < x + \delta \\
&\Longrightarrow x_1 = x - (x - x_1) < t < x + (x_2 - x) = x_2 \\
&\Longrightarrow f(x_1) \le f(t) \le f(x_2) \text{ (since } f \text{ is increasing)} \\
&\Longrightarrow f(x) - \epsilon < y_1 = f(x_1) \le f(t) \le f(x_2) = y_2 < f(x) + \epsilon \\
&\Longrightarrow |f(t) - f(x)| < \epsilon
\end{aligned}$$

4: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.

(a) Given $a, b \in \mathbb{R}$ with $a < b$, show that for every $x \in \mathbb{R}$ there is a point of the form $\big(t, f(t)\big)$ with $a \le t \le b$ which is nearest to the point $(x, 0)$.

Solution: We introduce some terminology and notation. Given a vector $u = (x, y) \in \mathbb{R}^2$, the the **length** of the vector is given by $\|u\| = \sqrt{x^2 + y^2}$. Given two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ in $\mathbb{R}^2$, the **distance** between the two points is $d(p_1, p_2) = \|p_2 - p_1\| = \big\|(x_2 - x_1, y_2 - y_1)\big\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Let $x \in \mathbb{R}$. Define $h : [a, b] \to [0, \infty)$ by $h(t) = d\big((t, f(t)), (x, 0)\big) = \sqrt{(t - x)^2 + f(t)^2}$. Since f is continuous, so is the function $g(t) = (t - x)^2 + f(t)^2$, and hence so is the function $h(t) = \sqrt{g(t)}$. By the Extreme Value Theorem, the continuous function h attains its minimum value on $[a, b]$, so we can choose $t_0 \in [a, b]$ so that $h(t_0) \le h(t)$ for every $t \in [a, b]$, so the distance $d\big((t_0, f(t_0), (x, 0)\big)$ is less than or equal to the distance $d\big((t, f(t)), (x, 0)\big)$ for every $t \in [a, b]$.

(b) Show that for every $x \in \mathbb{R}$ there is a point on the graph of f which is nearest to the point $(x, 0)$.

Solution: Let $x \in \mathbb{R}$. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(t) = d\big((t, f(t)), (x, 0)\big) = \sqrt{(t - x)^2 + f(t)^2}$. Note that since $h(t) = \sqrt{(t - x)^2 + f(t)^2} \ge \sqrt{(t - x)^2}$ for all $t \in \mathbb{R}$ and $\sqrt{(t - x)^2} \to \infty$ as $t \to \pm\infty$, it follows by the Comparison Theorem that $h(t) \to \infty$ as $t \to \pm\infty$. Choose $m \ge 0$ so that for all $t \in \mathbb{R}$ we have $t \ge m \implies h(t) \ge h(0)$ and $t \le -m \implies h(t) \ge h(0)$ (we can do this since $h(t) \to \infty$ as $t \to \pm\infty$). By Part (a), we can choose $t_0 \in [-m, m]$ so that $h(t_0) \le h(t)$ for all $t \in [-m, m]$. In particular, we have $h(t_0) \le h(0)$ because $0 \in [-m, m]$. Then $t \le -m \implies h(t) \ge h(0) \ge h(t_0)$ and $t \ge m \implies h(t) \ge h(0) \ge h(t_0)$ and $t \in [-m, m] \implies h(t) \ge h(t_0)$, and so we have $h(t_0) \le h(t)$ for all $t \in \mathbb{R}$.

(c) Define $g : \mathbb{R} \to [0, \infty)$ as follows. Given $x \in \mathbb{R}$, let $g(x)$ be the distance between the point $(x, 0)$ and a nearest point on the graph of f. Show that g is uniformly continuous.

Solution: Note that

$$g(x) = \min\left\{ d\big((t, f(t)), (x, 0)\big) \Big| t \in \mathbb{R} \right\}$$

and we know that the minimum exists by Part (b). We claim that g is uniformly continuous. Let $\epsilon > 0$. Choose $\delta = \epsilon$. Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$. Suppose that $|x - y| < \delta = \epsilon$. Choose $t_0 \in \mathbb{R}$ so that the point $(t_0, f(t_0))$ is a point on the graph which is nearest to $(x, 0)$ so that we have

$$g(x) = d\big((t_0, f(t_0), (x, 0)\big) = \Big\|(t_0, f(t_0)) - (x, 0)\Big\|$$

and

$$\begin{aligned} g(y) &= \min\left\{ d\big((t, f(t)), (y, 0)\big) \Big| t \in \mathbb{R} \right\} \le d\big((t_0, f(t_0)), (y, 0)\big) = \Big\|(t_0, f(t_0)) - (y, 0)\Big\| \\ &= \Big\|(t_0, f(t_0)) - (x, 0) + (x, 0) - (y_0)\Big\| \\ &\le \Big\|(t_0, f(t_0)) - (x, 0)\Big\| + \Big\|(x, 0) - (y, 0)\Big\| \quad \text{(by the triangle inequality)} \\ &= g(x) + |x - y| < g(x) + \epsilon. \end{aligned}$$

Similarly (by interchanging the roles of x and y) we have $g(x) < g(y) + \epsilon$ so that $|g(y) - g(x)| < \epsilon$.