AMATH/PMATH 331 Solutions to Assignment 1

1: (a) Let $A = \left\{\frac{2+(-1)^n}{n} \middle| n \in \mathbb{Z}^+\right\}$. Find (with proof) $\sup A$ and $\inf A$ and determine whether A has a maximum and whether A has a minimum.

Solution: Let $a_n = \frac{2+(-1)^n}{n}$ so that $S = \{a_n | n \in \mathbb{Z}^+\}$. We claim that $\sup S = \frac{3}{2}$. Since $\frac{3}{2} = a_2 \in S$ we must have $\sup S \geq \frac{3}{2}$. Also note that $\frac{3}{2}$ is an upper bound for S because $a_1 = 1 < \frac{3}{2}$ and $a_2 = \frac{3}{2}$ and for $n \geq 2$ we have $a_n = \frac{2+(-1)^n}{n} \leq \frac{3}{n} \leq \frac{3}{3} = 1 < \frac{3}{2}$. Since $\frac{3}{2}$ is an upper bound for S we must have $\sup S \leq \frac{3}{2}$. Thus $S = \frac{3}{2}$, as claimed. Note that since $\frac{3}{2} = a_2 \in S$, this is the maximum value of S.

We claim that $\inf S = 0$. Note that 0 is a lower bound for S because for all $n \geq 1$ we have $a_n = \frac{2+(-1)^n}{n} \geq \frac{1}{n} > 0$. Since 0 is a lower bound for S we have $\inf S \geq 0$. Suppose, for a contradiction, that $S > 0$, say $\inf S = m > 0$. Choose $n \in \mathbb{Z}^+$ with $n > \frac{3}{m}$ and note that we have $\frac{3}{n} < m$. Then $a_n = \frac{2+(-1)^n}{n} \leq \frac{3}{n} < m$, which contradicts the fact that m is a lower bound for S. Thus $\inf S = 0$, as claimed. Since $a_n > 0$ for all n, we have $0 \notin S$, and hence S does not have a minimum.

(b) For $F \subseteq \mathbb{R}$, we say that F is **dense** in $\mathbb{R}$ when for every $a, b \in \mathbb{R}$ with $a < b$ there exists $x \in F$ with $a < x < b$, and we say that F has the **supremum property** when for every nonempty set $A \subseteq F$, if A has an upper bound $c \in F$ then A has an upper bound $b \in F$ with $b \leq c$ for every upper bound $c \in F$ for A.

Prove that there is no proper subset of $\mathbb{R}$ which is dense in $\mathbb{R}$ and has the supremum property.

Solution: Let F be a proper subset of $\mathbb{R}$ which is dense in $\mathbb{R}$. Suppose, for a contradiction, that F has the supremum property. Choose $r \in \mathbb{R}$ with $r \notin F$. Let $A = \left\{x \in F \,\middle|\, x < r\right\}$. Note that A is non-empty because, since F is dense in $\mathbb{R}$, we can choose $a \in F$ with $r-1 < a < r$, and this value of x is in A. Also note that A is bounded above in F because, since F is dense in $\mathbb{R}$, we can choose $c \in F$ with $r < c < r+1$, and this value $c \in F$ is an upper bound for A (indeed, for all $x \in A$ we have $x < r < c$). Since A is nonempty and has an upper bound $c \in F$, and since we are assuming that F has the supremum property, we can choose an upper bound $b \in F$ for A with $b \leq c$ for every upper bound $c \in F$ for A.

We claim that since b is an upper bound for A, it follows that $b \geq r$. Suppose, for a contradiction, that $b < r$. Since F is dense in $\mathbb{R}$ we can choose $x \in F$ with $b < x < r$. Since $x \in F$ with $x < r$, we have $x \in A$. But then we have $x \in A$ with $x > b$ which contradicts the fact that b is an upper bound for A. Thus $b \geq r$, as claimed.

We claim that since $b \in F$ with $b \leq c$ for every upper bound $c \in F$ for A, it follows that $b < r$. Suppose, for a contradiction, that $r \leq b$. Since $b \in F$ and $r \notin F$ so that $r \neq b$, we have $r < b$. Since F is dense in $\mathbb{R}$ we can choose $c \in F$ with $r < c < b$. Since $r < c$, we have $x < r < c$ for every $x \in A$, and hence $c \in F$ is an upper bound for A. This contradicts the fact that $b \leq c$ for every upper bound $c \in F$ for A. Thus $b < r$, as claimed.

Thus $b \geq r$ and $b < r$, which gives the desired contradiction, so F does not have the supremum property.

2: Let $(x_n)_{n\geq p}$ and $(y_n)_{n\geq p}$ be sequences in $\mathbb{R}$ and let $a\in\mathbb{R}$.

(a) Use definitions of limits to prove that if $\lim\limits_{n\to\infty} x_n = a$ with $a>0$ and $\lim\limits_{n\to\infty} y_n=\infty$ then $\lim\limits_{n\to\infty}\dfrac{y_n}{x_n}=\infty$.

Solution: Suppose that $\lim\limits_{n\to\infty} x_n = a$ with $a>0$ and $\lim\limits_{n\to\infty} y_n=\infty$. Since $x_n\to a>0$ we can choose $m_1\in\mathbb{Z}_{\geq p}$ so that $n\geq m_1 \Longrightarrow |x_n-a|\leq\frac{a}{2}\Longrightarrow \frac{a}{2}\leq x_n\leq\frac{3a}{2}$. In particular, we remark that when $n\geq m_1$, we have $x_n>0$ so that $\frac{y_n}{x_n}$ is defined. Let $r\in\mathbb{R}$. Since $y_n\to\infty$, we can choose $m_2\in\mathbb{Z}_{\geq p}$ so that $n\geq m_2\Longrightarrow y_n\geq\frac{3a|r|}{2}$. Let $m=\max\{m_1,m_2\}$. Then for $n\geq m$ we have $x_n\leq\frac{3a}{2}$ and we have $y_n\geq\frac{3a|r|}{2}$, and so $\frac{y_n}{x_n}\geq\frac{3a|r|/2}{3a/2}=|r|\geq r$. Thus $\frac{y_n}{x_n}\to\infty$.

(b) Prove that for every sequence $(x_n)_{n\geq 1}$ of real numbers, if $\left|x_{n+1}-x_n\right|\leq\frac{1}{2^n}$ for all $n\in\mathbb{Z}^+$ then $(x_n)_{n\geq1}$ converges.

Solution: Let $(x_n)_{n\geq1}$ be a sequence of real numbers. Suppose that $\left|x_{n+1}-x_n\right|\leq\frac{1}{2^n}$ for all $n\in\mathbb{Z}^+$. We shall show that $(x_n)_{n\geq1}$ is Cauchy. Let $\epsilon>0$. Choose $m\in\mathbb{Z}^+$ large enough so that $\frac{1}{2^{m-1}}<\epsilon$ $\big($equivalently, choose $m\in\mathbb{Z}^+$ with $m>1+\ln(1/\epsilon)/\ln 2\big)$. Let $k,\ell\geq m$. If $k=\ell$ then $|a_k-a_\ell|=0<\epsilon$. Suppose that $k\neq\ell$, say $k<\ell$ (the case $\ell<k$ is similar). Then

$$\begin{aligned}|a_k-a_\ell| &= |a_\ell-a_k| = \left|a_\ell-a_{\ell-1}+a_{\ell-1}-a_{\ell-2}+a_{\ell-2}-\cdots-a_{k+1}-a_{k+1}-a_k\right|\\ &\leq |a_\ell-a_{\ell-1}|+|a_{\ell-1}+a_{\ell-2}|+\cdots+|a_{k+1}-a_k|\\ &\leq \tfrac{1}{2^{\ell-1}}+\tfrac{1}{2^{\ell-2}}+\cdots+\tfrac{1}{2^{k+1}}+\tfrac{1}{2^k}=2\big(\tfrac{1}{2^k}-\tfrac{1}{2^\ell}\big)<\tfrac{1}{2^{k-1}}\leq\tfrac{1}{2^{m-1}}<\epsilon.\end{aligned}$$

Thus $(a_n)_{n\geq1}$ is Cauchy, as claimed, and so it converges by the Cauchy Criterion for Convergence.

3: (a) Use the definition of the limit to prove that $\displaystyle\lim_{x\to -1}\frac{x+1}{x^2-1}=-\frac{1}{2}$.

Solution: Note that for $x\neq\pm1$ we have

$$\left|\frac{x+1}{x^2-1}+\frac{1}{2}\right|=\left|\frac{x+1}{(x-1)(x+1)}+\frac{1}{2}\right|=\left|\frac{1}{x-1}+\frac{1}{2}\right|=\left|\frac{2+(x-1)}{2(x-1)}\right|=\frac{|x+1|}{2|x-1|}\,.$$

Also note that

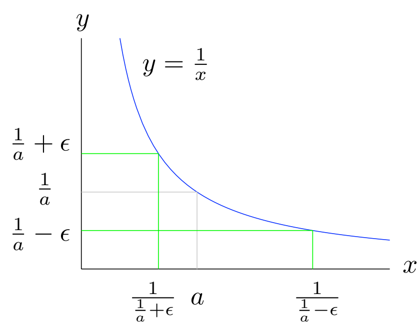
$$|x+1|<1\implies -2<x<0\implies -3<x-1<-1\implies 1<|x-1|<3\implies \frac{1}{3}<\frac{1}{|x-1|}<1\,.$$

Given $\epsilon>0$ we choose $\delta=\min\big(1,2\epsilon\big)$ and then we have

$$\begin{aligned}0<|x+1|<\delta&\implies\big(x\neq\pm1\text{ and }|x+1|<1\text{ and }|x+1|<2\epsilon\big)\\&\implies\big(x\neq\pm1\text{ and }\tfrac{1}{|x-1|}<1\text{ and }|x+1|<2\epsilon\big)\\&\implies\left|\frac{x+1}{x^2-1}+\frac{1}{2}\right|=\frac{|x+1|}{2|x-1|}<\frac{2\epsilon}{2\cdot1}=\epsilon\,.\end{aligned}$$

(b) Let $a>0$ and let $0<\epsilon<\frac{1}{a}$. Find (with proof) the *largest* value of $\delta>0$ with the property that for all x with $0<|x-a|<\delta$ we have $\left|\frac{1}{x}-\frac{1}{a}\right|<\epsilon$.

Solution: Careful consideration of the following graph



leads us to guess that the largest such value δ is $\delta=a-\frac{1}{\frac{1}{a}+\epsilon}$. To verify this algebraically, we first show that $a-\frac{1}{\frac{1}{a}+\epsilon}<\frac{1}{\frac{1}{a}-\epsilon}-a$ (as the graph suggests). Note that $a-\frac{1}{\frac{1}{a}+\epsilon}=a-\frac{a}{1+a\epsilon}=\frac{a+a^2\epsilon-a}{1+a\epsilon}=\frac{a^2\epsilon}{1+a\epsilon}$ and similarly $\frac{1}{\frac{1}{a}-\epsilon}-a=\frac{a^2\epsilon}{1-a\epsilon}$. Since a and ϵ are positive we have

$$-a\epsilon<a\epsilon\implies 1-a\epsilon<1+a\epsilon\implies\frac{1}{1+a\epsilon}<\frac{1}{1-a\epsilon}\implies\frac{a^2\epsilon}{1+a\epsilon}<\frac{a^2\epsilon}{1-a\epsilon}\implies a-\frac{1}{\frac{1}{a}+\epsilon}<\frac{1}{\frac{1}{a}-\epsilon}-a\,,$$

as claimed. It follows that when $\delta=a-\frac{1}{\frac{1}{a}+\epsilon}$ we have

$$\begin{aligned}|x-a|<\delta&\implies a-\delta<x<a+\delta\implies a-\left(a-\frac{1}{\frac{1}{a}+\epsilon}\right)<x<a+\left(\frac{1}{\frac{1}{a}-\epsilon}-a\right)\\&\implies\frac{1}{\frac{1}{a}+\epsilon}<x<\frac{1}{\frac{1}{a}-\epsilon}\implies\frac{1}{a}-\epsilon<\frac{1}{x}<\frac{1}{a}+\epsilon\implies\left|\frac{1}{x}-\frac{1}{a}\right|<\epsilon\,.\end{aligned}$$

On the other hand, when $\delta>a-\frac{1}{\frac{1}{a}+\epsilon}$ we can choose x with $a-\delta<x<\frac{1}{\frac{1}{a}+\epsilon}=\frac{a}{1+a\epsilon}$ but then we have $a-\delta<x<a$ so that $0<|x-a|<\delta$, and we also have $\frac{1}{x}>\frac{1}{a}+\epsilon$ so that $\left|\frac{1}{x}-\frac{1}{a}\right|>\epsilon$.

4: For any sets A and B, the **set difference** of A and B is the set $A \setminus B = \{x \in A \mid x \notin B\}$.

For any non-empty sets X and Y, we say that f is a **function** from X to Y, and we write $f : X \to Y$, when for every $x \in X$ there exists a unique corresponding element $f(x) \in Y$. The set X is called the **domain** of f and the set $f(X) = \{f(x) \mid x \in X\}$ is called the **range** of f.

Let X and Y be nonempty sets and let $f : X \to Y$. When $A \subseteq X$, the **image** of A under f is the set $f(A) = \{f(x) \mid x \in A\}$. When $C \subseteq Y$, the **inverse image** of C under f is the set $f^{-1}(C) = \{x \in X \mid f(x) \in C\}$. We say that f is **injective** (or **one-to-one**, written as 1:1) when f has the property that for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Prove that the following are equivalent.

(1) f is injective.
(2) $f(A \cap B) = f(A) \cap f(B)$ for all subsets $A, B \subseteq X$.
(3) $f(A \setminus B) = f(A) \setminus f(B)$ for all subsets $A, B \subseteq X$.
(4) $f^{-1}\big(f(A)\big) = A$ for all subsets $A \subseteq X$.

Solution: First we show that (1) $\iff$ (2). Suppose that f is injective. Let $A, B \subseteq X$. Let $y \in f(A \cap B)$. Choose $x \in A \cap B$ with $f(x) = y$. Since $x \in A$ and $y = f(x)$ we have $y \in f(A)$. Since $x \in B$ and $y = f(x)$ we have $y \in f(B)$. Thus $y \in f(A) \cap f(B)$. This proves that $f(A \cap B) \subseteq f(A) \cap f(B)$ (we did not use the fact that f was injective). Now let $y \in f(A) \cap f(B)$. Since $y \in f(A)$ we can choose $x_1 \in A$ with $f(x_1) = y$. Since $y \in f(B)$ we can choose $x_2 \in B$ with $f(x_2) = y$. Since $f(x_1) = y = f(x_2)$ and f is injective, we must have $x_1 = x_2$, say $x_1 = x_2 = x$. Since $x = x_1 \in A$ and $x = x_2 \in B$ we have $x \in A \cap B$. Since $x \in A \cap B$ and $y = f(x_1) = f(x_2) = f(x)$ we have $y \in f(A \cap B)$. This shows that $f(A) \cap f(B) \subseteq f(A \cap B)$, and completes the proof that (1) $\implies$ (2). Conversely, suppose that f is not injective. Choose $x_1, x_2 \in X$ with $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$, and let $y = f(x_1) = f(x_2)$. Let $A = \{x_1\}$ and $B = \{x_2\}$. Then $f(A) \cap f(B) = \{y\} \cap \{y\} = \{y\}$ but $A \cap B = \{x_1\} \cap \{x_2\} = \emptyset$ so $f(A \cap B) = f(\emptyset) = \emptyset$. Thus (2) $\implies$ (1).

Next we show that (1) $\iff$ (3). Suppose that f is injective. Let $A, B \subseteq X$. Let $y \in f(A \setminus B)$. Choose $x_1 \in A \setminus B$ with $f(x_1) = y$. Since $x_1 \in A$ and $f(x_1) = y$ we have $y \in f(A)$. Suppose, for a contradiction, that $y \in f(B)$. Choose $x_2 \in B$ with $f(x_2) = y$. Since $f(x_1) = y = f(x_2)$ and f is injective, we must have $x_1 = x_2$. This is not possible since $x_1 \notin B$ and $x_2 \in B$. Thus we must have $y \notin f(B)$. Since $y \in f(A)$ and $y \notin f(B)$, we have $y \in f(A) \setminus f(B)$. This proves that $f(A \setminus B) \subseteq f(A) \setminus f(B)$. Now let $y \in f(A) \setminus f(B)$. Since $y \in f(A)$ we can choose $x \in A$ so that $f(x) = y$. Since $y \notin f(B)$ and $y = f(x)$, we cannot have $x \in B$. Since $x \in A$ and $x \notin B$ we have $x \in A \setminus B$. Since $x \in A \setminus B$ and $y = f(x)$ we have $y \in f(A \setminus B)$. This shows that $f(A) \setminus f(B) \subseteq f(A \setminus b)$ and completes the proof that (1) $\implies$ (3). Conversely, suppose that f is not injective. Choose $x_1, x_2 \in X$ with $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$ and let $y = f(x_1) = f(x_2)$. Let $A = \{x_1, x_2\}$ and $B = \{x_2\}$. Then $A \setminus B = \{x_1\}$ so $f(A \setminus B) = f(\{x_1\}) = \{y\}$ but $f(A) = \{f(x_1), f(x_2)\} = \{y\} = f(B)$ so that $f(A) \setminus f(B) = \emptyset$. Thus (3) $\implies$ (1).

Finally, we show that (1) $\iff$ (4). Suppose that f is injective. Let $A \subseteq X$. Let $x_1 \in f^{-1}\big(f(A)\big)$. This means that we have $f(x_1) \in f(A)$. Since $f(x_1) \in f(A)$, we can choose $x_2 \in A$ so that $f(x_1) = f(x_2)$. Since $f(x_1) = f(x_2)$ and f is injective, we must have $x_1 = x_2$. Thus $x_1 = x_2 \in A$. This proves that $f^{-1}\big(f(A)\big) \subseteq A$. Now let $x \in A$, and let $y = f(x)$. Since $y = f(x)$ with $x \in A$, we have $y \in f(A)$. Since $f(x) = y$ with $y \in f(A)$, we have $x \in f^{-1}\big(f(A)\big)$. This proves that $A \subseteq f^{-1}\big(f(A)\big)$, and completes the proof that (1) $\implies$ (4). Conversely, suppose that f is not injective. Choose $x_1, x_2 \in X$ with $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$ and let $y = f(x_1) = f(x_2)$. Let $A = \{x_1\}$. Then $f(A) = \{y\}$ so we have $f^{-1}\big(f(A)\big) = f^{-1}(\{y\}) = \{x \in X \mid f(x) = y\}$. Since $f(x_2) = y$ we have $x_2 \in f^{-1}\big(f(A)\big)$. Since $x_2 \in f^{-1}\big(f(A)\big)$ but $x_2 \notin \{x_1\} = A$, we see that $f^{-1}\big(f(A)\big) \neq A$. Thus (4) $\implies$ (1).