AMATH/PMATH 331 Assignment 1 Due Sept 23, 11:00 pm

1: (a) Let $A = \left\{ \frac{2+(-1)^n}{n} \middle| n \in \mathbb{Z}^+ \right\}$. Find (with proof) $\sup A$ and $\inf A$ and determine whether A has a maximum and whether A has a minimum.

(b) For $F \subseteq \mathbb{R}$, we say that F is **dense** in $\mathbb{R}$ when for every $a, b \in \mathbb{R}$ with $a < b$ there exists $x \in F$ with $a < x < b$, and we say that F has the **supremum property** when for every nonempty set $A \subseteq F$, if A has an upper bound $c \in F$ then A has an upper bound $b \in F$ with $b \leq c$ for every upper bound $c \in F$ for A.

Prove that there is no proper subset of $\mathbb{R}$ which is dense in $\mathbb{R}$ and has the supremum property.

2: Let $(x_n)_{n \geq p}$ and $(y_n)_{n \geq p}$ be sequences in $\mathbb{R}$ and let $a \in \mathbb{R}$.

(a) Use definitions of limits to prove that if $\lim\limits_{n \to \infty} x_n = a$ with $a > 0$ and $\lim\limits_{n \to \infty} y_n = \infty$ then $\lim\limits_{n \to \infty} \dfrac{y_n}{x_n} = \infty$.

(b) Prove that for every sequence $(x_n)_{n \geq 1}$ of real numbers, if $\left| x_{n+1} - x_n \right| \leq \frac{1}{2^n}$ for all $n \in \mathbb{Z}^+$ then $(x_n)_{n \geq 1}$ converges.

3: (a) Use the definition of the limit to prove that $\lim\limits_{x \to -1} \dfrac{x+1}{x^2 - 1} = -\frac{1}{2}$.

(b) Let $a > 0$ and let $0 < \epsilon < \frac{1}{a}$. Find (with proof) the *largest* value of $\delta > 0$ with the property that for all x with $0 < |x - a| < \delta$ we have $\left| \frac{1}{x} - \frac{1}{a} \right| < \epsilon$.

4: For any sets A and B, the **set difference** of A and B is the set $A \setminus B = \left\{ x \in A \,\middle|\, x \notin B \right\}$.

For any non-empty sets X and Y, we say that f is a **function** from X to Y, and we write $f : X \to Y$, when for every $x \in X$ there exists a unique corresponding element $f(x) \in Y$. The set X is called the **domain** of f and the set $f(X) = \left\{ f(x) \,\middle|\, x \in X \right\}$ is called the **range** of f.

Let X and Y be nonempty sets and let $f : X \to Y$. When $A \subseteq X$, the **image** of A under f is the set $f(A) = \left\{ f(x) \,\middle|\, x \in A \right\}$. When $C \subseteq Y$, the **inverse image** of C under f is the set $f^{-1}(C) = \left\{ x \in X \,\middle|\, f(x) \in C \right\}$.

We say that f is **injective** (or **one-to-one**, written as 1:1) when f has the property that for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Prove that the following are equivalent.

(1) f is injective.
(2) $f(A \cap B) = f(A) \cap f(B)$ for all subsets $A, B \subseteq X$.
(3) $f(A \setminus B) = f(A) \setminus f(B)$ for all subsets $A, B \subseteq X$.
(4) $f^{-1}\big(f(A)\big) = A$ for all subsets $A \subseteq X$.