SYDE 311 Advanced Engineering Math 2, Solutions to the Exam, Spring 2024

[10] **1:** (a) For the pair of ODEs $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x^2+y-1 \\ 2y-xy \end{pmatrix}$, find all the equilibrium points.

Solution: A point (x,y) is an equilibrium point when $y = 1 - x^2$ (1) and $y(2-x) = 0$ (2). From (2) we find that $y = 0$ or $x = 2$. When $y = 0$, equation (1) gives $x^2 = 1$ so that $x = \pm 1$, and when $x = 2$ equation (1) gives $y = 1 - 2^2 = -3$. Thus the equilibrium points are $(x,y) = (\pm 1, 0), (2, -3)$.

(b) For $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} xy+x \\ y^2-x \end{pmatrix}$, determine whether $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is attracting or repelling.

Solution: Let $F(x,y) = \begin{pmatrix} xy+x \\ y^2-x \end{pmatrix}$. Then $DF(x,y) = \begin{pmatrix} y+1 & x \\ -1 & 2y \end{pmatrix}$. Let $A = DF(1,-1) = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$. The characteristic polynomial is $g(r) = r^2 + 2r + 1 = (r+1)^2$. The only eigenvalue is $r = -1$, which is negative, so the equilibrium point $(1,-1)$ is attracting.

(c) For $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} xy^2 \\ yx-y \end{pmatrix}$, find a conserved quantity $H(x,y)$.

Solution: We need to solve the DE $\frac{dy}{dx} = \frac{y'}{x'} = \frac{yx-y}{xy^2} = \frac{x-1}{xy}$. The DE is separable, so we write it as $y\,dy = \frac{x-1}{x}\,dx$ and integrate both sides to get $\frac{1}{2}y^2 = \int 1 - \frac{1}{x}\,dx = x - \ln x + c$. Thus $H(x,y) = \frac{1}{2}y^2 - x + \ln x$ is a conserved quantity.

[10] **2:** (a) Find the 4^{th} Taylor polynomial at 0 for the solution to the IVP $y'+xy=1+x^2$ with $y(0)=2$.

Solution: Let $y=c_0+c_1x+c_2x^2+c_3x^3+\cdots$ and $y'=c_1+2c_2x+3c_3x^2+\cdots$. Put this in the DE to get

$$\left(c_1+2c_2x+3c_3c^2+4c_4x^3+\cdots\right)+\left(c_0x+c_1x^2+c_2x^3+\cdots\right)=1+x.$$

To get $y(0)=2$ we need $c_0=2$, and then we equate coefficients of x^n in the above equation: By equating coefficients of x^0 we get $c_1=1$. From the coefficients of x^1 we get $2c_2+c_0=0$ so that $c_2=-\frac{1}{2}c_0=-\frac{1}{2}\cdot 2=-1$. From x^2 we get $3c_3+c_1=1$ so that $c_3=\frac{1}{3}(1-c_1)=\frac{1}{3}(1-1)=0$. From x^3 we get $4c_4+c_2=0$ so that $c_4=-\frac{1}{4}c_2=-\frac{1}{4}(-1)=\frac{1}{4}$. Thus the 4^{th} Taylor polynomial is

$$y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4=2+x-\tfrac{1}{2}x^2+\tfrac{1}{4}x^4.$$

(b) Consider the ODE $x^2y''-x^2y'+(x-2)\,y=0$. Following Frobenius' method, find two values of r such that the DE has a solution of the form $y=\sum_{n=0}^{\infty}c_nx^{n+r}$, then find an exact, closed form formula for a solution with $r=-1$ and $c_0=1$.

Solution: Let $y=\sum\limits_{n\geq 0}c_nx^{n+r}$, $y'=\sum\limits_{n\geq 0}(n+r)c_nx^{n+r-1}$ and $y''=\sum\limits_{n\geq 0}(n+r)(n+r-1)c_nx^{n+r-2}$. Put these in the DE to get

$$\begin{aligned}
0&=\sum_{n\geq 0}(n+r)(n+r-1)c_nx^{n+r}-\sum_{n\geq 0}(n+r)c_nx^{n+r+1}+\sum_{n\geq 0}c_nx^{n+r+1}-\sum_{n\geq 0}2c_nx^{n+r}\\
&=x^r\Big(\sum_{m\geq 0}(m+r)(m+r-1)c_mx^m-\sum_{m\geq 1}(m+r-1)c_{m-1}x^m+\sum_{m\geq 1}c_{m-1}x^m-\sum_{m\geq}2c_mx^m\Big)\\
0&=\sum_{m\geq 0}\big((m+r)(m+r-1)-2\big)c_mx^m-\sum_{m\geq 1}(m+r-2)c_{m-1}x^m
\end{aligned}$$

When $m=0$, if $c_0\neq 0$ then equationg the coefficient of x^0 gives $r(r-1)-2=0$, that is $r^2-r-2=0$, so that $(r-2)(r+1)=0$. To get a non-zero solution we need $r=2$ or $r=-1$.

When $r=-1$ and $c_0=1$, equating the coefficient of x_m, $m\geq 1$ gives $((m-1)(m-2)-2)c_m-(m-3)c_{m-1}=0$ so that

$$c_m=\tfrac{m-3}{(m-1)(m-2)-2}c_{m-1}=\tfrac{m-3}{m^2-3m}c_{m-1}=\tfrac{1}{m}c_{m-1}.$$

Thus $c_n=\frac{1}{n!}$ for all $n\geq 0$, and the solution is $y=x^{-1}\sum\limits_{n\geq 0}\frac{1}{n!}\,x^n=\frac{1}{x}\,e^x$.

[10] **3:** (a) Find the Fourier series for the 2π-periodic function f with $f(x)=|x|$ for $-\pi\leq x\leq\pi$.

Solution: Since f is even we have $b_n = 0$ for all $n \geq 0$ and we have

$$\begin{aligned} a_0 &= \tfrac{1}{\pi}\int_0^\pi x\,dx = \tfrac{1}{\pi}\left[\tfrac{1}{2}x^2\right]_0^\pi = \tfrac{\pi}{2} \\ a_n &= \tfrac{2}{\pi}\int_0^\pi x\cos nx\,dx = \tfrac{2}{\pi}\left(\left[\tfrac{1}{n}x\sin nx\right]_0^\pi - \int_0^\pi \tfrac{1}{n}\sin nx\,dx\right) \\ &= \tfrac{2}{\pi}\left(0 + \left[\tfrac{1}{n^2}\cos nx\right]_0^\pi\right) = \tfrac{2}{\pi n^2}((-1)^n - 1) = \left\{\begin{array}{cl} 0 & \text{, if } n \text{ is even} \\ -\frac{4}{\pi n^2} & \text{, if } n \text{ id odd} \end{array}\right\}. \end{aligned}$$

Thus the Fourier series of f is given by $f(x) = \frac{\pi}{2} - \sum\limits_{n\text{ odd}} \frac{4}{\pi n^2}\cos nx$.

(b) Let f be the 2π-periodic function with $f(x) = -1$ for $-\pi \leq x < 0$ and $f(x) = 1$ for $0 \leq x < \pi$. The Fourier series of f is given by $\sum\limits_{n\text{ odd}} \frac{4}{\pi n}\sin nx$. By evaluating at $x = \frac{\pi}{2}$, and by using Parceval's identity, find $R=\sum\limits_{k=0}^\infty \frac{(-1)^k}{2k+1}$, $S=\sum\limits_{k=0}^\infty \frac{1}{(2k+1)^2}$, and $T=\sum\limits_{n=1}^\infty \frac{1}{n^2}$.

Solution: Note that $\sin\frac{n\pi}{2} = 0$ when n is even and $\sin\frac{(2k+1)\pi}{2} = (-1)^k$, so evaluating at $x = \frac{\pi}{2}$ gives

$$1 = f\left(\tfrac{\pi}{2}\right) = \sum_{n\text{ odd}} \tfrac{4}{\pi n}\sin\tfrac{n\pi}{2} = \tfrac{4}{\pi}\sum_{k\geq 0}\tfrac{1}{(2k+1)}\sin\tfrac{(2k+1)\pi}{2} = \tfrac{4}{\pi}\sum_{k\geq 0}\tfrac{(-1)^k}{2k+1} = \tfrac{4}{\pi}R.$$

Thus we find that $R = \frac{\pi}{4}$. Next, note that

$$\|f\|^2 = \int_{-\pi}^{\pi} f(x)^2\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi.$$

so, by Parseval's identity, we have

$$2\pi = \|f\|^2 = \pi\sum_{n\text{ odd}}\left(\tfrac{4}{\pi n}\right)^2 = \tfrac{16}{\pi}S$$

so that $S = \frac{2\pi}{16/\pi} = \frac{\pi^2}{8}$. Finally, notice that we can write $\sum\limits_{n\text{ even}}\frac{1}{n^2} = \sum\limits_{k\geq 1}\frac{1}{(2k)^2} = \frac{1}{4}\sum\limits_{k\geq 1}\frac{1}{k^2} = T$ so we have $T = \sum\limits_{n\text{ even}}\frac{1}{n^2} + \sum\limits_{n\text{ odd}}\frac{1}{n^2} = \frac{1}{4}T + S$ so that $S = \frac{3}{4}T$, and hence $T = \frac{4}{3}S = \frac{4}{3}\frac{\pi^2}{8} = \frac{\pi^2}{6}$.

[10] **4:** (a) Find the solution $u(x,y)$ to Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for $0 \le x \le 2$ and $0 \le y \le 1$ which satisfies the boundary conditions $u(x,0) = x$, $u(x,1) = 1$, $u(0,y) = y$ and $u(2,y) = 2 - y$.

Solution: We choose $u(x,y) = x + y - xy$. Then u satisfies Laplace's equation and we have $u(x,0) = x$, $u(x,1) = x + 1 - x = 1$, $u(0,y) = y$ and $u(2,y) = 2 + y - 2y = 2 - y$.

(b) Find the solution $u = u(x,t)$ to the heat equation $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ for $0 \le x \le 2$ and $t \ge 0$, satisfying the insulated ends condition $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(2,t) = 0$ for all $t \ge 0$ and the initial condition $u(x,0) = 2\cos^2(\pi x)$ for all $0 \le x \le 2$.

Solution: Taking $c = 2$, $\ell = 2$, and $f(x) = 2\cos^2(\pi x) = 1 + \cos(2\pi x)$ for $0 \le x \le 2$, the desired solution $u = u(x,t)$ to Laplace's equation is given by

$$u(x,t) = \sum_{n \ge 0} a_n e^{-(cn\pi/\ell)^2 t} \cos\left(\tfrac{n\pi}{\ell}x\right) = \sum_{n \ge 0} a_n e^{-(n\pi)^2 t} \cos\left(\tfrac{n\pi}{2}x\right)$$

where the constants a_n are the Fourier coefficients for the even 2ℓ-periodic (that is 4-periodic) function given by $f(x) = 1 + \cos(2\pi x)$ for $0 \le x \le \pi$, namely $a_0 = 1$ and $a_4 = 1$ and $a_n = 0$ for $n \ne 0, 4$. The solution is

$$u(x,t) = 1 + e^{-(4\pi)^2 t} \cos(2\pi x).$$

(c) Find all negative values $k < 0$ for which there exists a non-zero solution to the ODE $y'' = ky$ for $y = y(x)$ with $y(0) = 0$ and $y'(2) = 0$.

Solution: Let $k < 0$, say $k = -\sigma^2$ with $\sigma > 0$. The DE becomes $y'' + \sigma^2 y = 0$ which has solutions $y = y(x) = a\sin(\sigma x) + b\cos(\sigma x)$. To get $y(0) = 0$ we need $b = 0$ so that $y(x) = a\sin(\sigma x)$ and hence $y'(x) = \sigma a\cos(\sigma x)$. To get $y'(2) = 0$ we need $\sigma a\cos(2\sigma) = 0$. When $a = 0$ we get the zero solution, so for a non-zero solution we need $\cos(2\sigma) = 0$, which occurs when $2\sigma = \frac{\pi}{2} + \pi n$, that is when $\sigma = \frac{\pi}{4} + \frac{n\pi}{2}$, for some integer $n \ge 0$. Thus we get non-zero solutions when $k = -\sigma^2 = -\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)^2$ for some $0 \le n \in \mathbb{Z}$. For this value of k, we remark that the solution is given by $y = y_n(x) = a_n \sin\left(\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)x\right)$.

[10] **5:** (a) Use Euler's method with step size $h = \Delta t = \frac{1}{2}$ to approximate the point $(x(1), y(1))$ when $(x(t), y(t))$ is the solution to $x' = y^2 - 1$ and $y' = x + y$ with $x(0) = y(0) = 0$.

Solution: We obtain

k	t_k	x_k	y_k	y_k^2-1	x_k+y_k
0	0	0	0	-1	0
1	$\frac{1}{2}$	$-\frac{1}{2}$	0	-1	$-\frac{1}{2}$
2	1	-1	$-\frac{1}{4}$		

Thus $(x(1), y(1)) \cong (x_2, y_2) = \left(-1, -\frac{1}{4}\right)$.

(b) Find the first approximation $\binom{x_1}{y_1}$ when Newton's method is used to approximate a solution to the equation $\binom{2+x-y^2}{2+y-x^2} = \binom{0}{0}$ starting with $\binom{x_0}{y_0} = \binom{1}{-1}$.

Solution: Let $F\binom{x}{y} = \begin{pmatrix} 2+x-y^2 \\ 2+y-x^2 \end{pmatrix}$. Then

$$DF\binom{x}{y} = \begin{pmatrix} 2 & -2y \\ -2x & 1 \end{pmatrix} \ , \ F\binom{1}{-1} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \ , \ DF\binom{1}{-1} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \ , \ DF\binom{1}{-1}^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

and so

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = F\binom{1}{-1} - DF\binom{1}{-1}^{-1} F\binom{1}{-1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 \\ -9 \end{pmatrix}.$$

(c) Find the weights w_0, w_1 and w_2 for the Newton-Cotes quadrature rule using the points $x_0 = 0$, $x_1 = 1$ and $x_2 = 3$ in the interval $[0, 3]$ to give $\int_0^3 f(x)\,dx \cong \sum_{k=0}^{2} w_k f(x_k)$.

Solution: First we let $g_0(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}(x-1)(x-3)$, and $g_1(x) = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}x(x-3)$ and $g_2(x) = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}x(x-1)$, and then we choose

$$\begin{aligned}
w_0 &= \int_0^3 g_0(x)\,dx = \tfrac{1}{3}\int_0^3 x^2 - 4x + 3 \ dx = \tfrac{1}{3}\left[\tfrac{1}{3}x^3 - 2x^2 + 3x\right]_0^3 = 3 - 6 + 3 = 0, \\
w_1 &= \int_0^3 g_1(x)\,dx = -\tfrac{1}{2}\int_0^3 x^2 - 3x \ dx = -\tfrac{1}{2}\left[\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2\right]_0^3 = -\tfrac{9}{2}\left(1 - \tfrac{3}{2}\right) = \tfrac{9}{4} \text{ , and} \\
w_2 &= \int_0^3 g_2(x)\,dx = \tfrac{1}{6}\int_0^3 x^2 - x \ dx = \tfrac{1}{6}\left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right]_0^3 = \tfrac{9}{6}\left(1 - \tfrac{1}{2}\right) = \tfrac{3}{4}.
\end{aligned}$$

For an alternate solution, we require that $\sum_{k=0}^{2} w_k p(x_k) = \int_0^3 p(x)\,dx$ for each $p(x) \in \{1, x, x^2\}$. Taking $p(x) = 1$ gives $w_0 + w_1 + w_2 = \int_0^3 1 \ dx = 3$ (1), taking $p(x) = x$ gives $0w_0 + 1w_1 + 3w_3 = \int_0^3 x \ dx = \frac{9}{2}$ (2) and taking $p(x) = x^2$ gives $0w_0 + 1w_1 + 9w_2 = \int_0^3 x^2\,dx = 9$ (3). We solve these 3 equations:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & \frac{9}{2} \\ 0 & 1 & 9 & 9 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & -\frac{3}{2} \\ 0 & 1 & 3 & \frac{9}{2} \\ 0 & 0- & 6 & \frac{9}{2} \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{9}{4} \\ 0 & 0 & 1 & \frac{3}{4} \end{array}\right)$$

Thus, as in the first solution, we take $w_0 = 0$, $w_1 = \frac{9}{4}$ and $w_2 = \frac{3}{4}$.