SYDE Advanced Math 2, Solutions to Assignment 9

1: Find the Fourier series for the 2π-periodic function $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=x$ when $-\pi\le x<\pi$, then use Parseval's Identity to show that $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

Solution: We have

$$a_0=\tfrac{1}{2\pi}\int_{-\pi}^{\pi}x\,dx=0\text{ , and}$$

$$a_n=\tfrac{1}{\pi}\int_{-\pi}^{\pi}x\cos nx\,dx=0,$$

and Integration by Parts gives

$$b_n=\tfrac{1}{\pi}\int_{-\pi}^{\pi}x\sin nx\,dx=\tfrac{1}{\pi}\left(\Big[-\tfrac{1}{n}\,x\cos nx\Big]_{-\pi}^{\pi}+\int_{-\pi}^{\pi}\tfrac{1}{n}\cos nx\,dx\right)=\tfrac{1}{\pi}\big(\tfrac{(-1)^{n+1}2\pi}{n}+0\big)=\tfrac{2(-1)^{n+1}}{n}.$$

Thus the Fourier series of f is

$$s(f)(x)=\sum_{n=1}^{\infty}\tfrac{2(-1)^{n+1}}{n}\sin nx.$$

By Parseval's Identity, we have

$$\|f\|_2{}^2=2\pi\,a_0(f)+\pi\sum_{n=1}^{\infty}a_n(f)^2+\pi\sum_{n=1}^{\infty}b_n(f)=\pi\sum_{n=1}^{\infty}\tfrac{4}{n^2}=4\pi\sum_{n=1}^{\infty}\tfrac{1}{n^2}.$$

On the other hand,

$$\|f\|_2{}^2=\int_{-\pi}^{\pi}f(x)^2\,dx=\int_{-\pi}^{\pi}x(x)^2\,dx=\tfrac{2\pi^3}{3}$$

so we have $4\pi\sum\limits_{n=1}^{\infty}\frac{1}{n^2}=\frac{2\pi^3}{3}$ and hence $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

2: Let $f:\mathbb{R}\to\mathbb{R}$ be the function of period 4 given by $f(x)=1$ for $-1\le x<1$ and $f(x)=0$ for $1\le x<3$. Find the Fourier series for f, then evaluate at $x=0$ to find $\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$.

Solution: Since $f(x)$ is even (except at the points of discontinuity $x\!=\!1\!+\!2k$, $k\!\in\!\mathbb{Z}$) we have $b_n\!=\!0$ for all n, and we have

$$a_0=\tfrac{1}{4}\int_{-2}^{2}f(x)\,dx=\tfrac{1}{4}\int_{-1}^{1}1\,dx=\tfrac{1}{2}\quad\text{, and}$$

$$a_n=\tfrac{1}{2}\int_{-2}^{2}f(x)\cos\left(\tfrac{\pi n}{2}\,x\right)dx=\tfrac{1}{2}\int_{-1}^{1}\cos\left(\tfrac{n\pi}{2}\,x\right)dx=\int_{0}^{1}\cos\left(\tfrac{n\pi}{2}\,x\right)dx=\left[\tfrac{2}{n\pi}\sin\left(\tfrac{n\pi}{2}\,x\right)\right]_0^1$$

$$=\left\{\begin{array}{c l} 0 & \text{, if } n \text{ is even} \\ \frac{2}{n\pi} & \text{, if } n=1\!+\!4k\,,\ k\!\in\!\mathbb{Z} \\ -\frac{2}{n\pi} & \text{, if } n=3\!+\!4k\,,\ k\!\in\!\mathbb{Z}\end{array}\right\}.$$

Since $f(x)$ is equal to the sum of its Fourier series (except at the points of discontinuity), we have

$$\begin{aligned} f(x)&=\tfrac{1}{2}+\tfrac{2}{\pi}\left(\tfrac{1}{1}\cos\left(\tfrac{\pi}{2}\,x\right)-\tfrac{1}{3}\cos\left(\tfrac{3\pi}{2}x\right)+\tfrac{1}{5}\cos\left(\tfrac{5\pi}{2}x\right)-\tfrac{1}{7}\cos\left(\tfrac{7\pi}{2}x\right)+\cdots\right)\\ &=\tfrac{1}{2}+\sum_{k=0}^{\infty}\tfrac{(-1)^k}{(2k+1)}\cos\left(\tfrac{(2k+1)\pi}{2}x\right).\end{aligned}$$

Evaluating at $x=0$ gives

$$1=f(0)=\tfrac{1}{2}+\tfrac{2}{\pi}\left(\tfrac{1}{1}-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\cdots\right)=\tfrac{1}{2}+\tfrac{2}{\pi}\sum_{k=0}^{\infty}\tfrac{(-1)^k}{2k+1}$$

so we have $\sum\limits_{k=0}^{\infty}\frac{(-1)^k}{2k+1}=\frac{\pi}{4}$.

3: Let $f:\mathbb{R}\to\mathbb{R}$ be the 2π-periodic function with $f(x)=x^3-\pi^2x$ for $-\pi\le x\le\pi$. Find the Fourier series for f, then evaluate at $x=\frac{\pi}{2}$ to find $\sum\limits_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}$.

Solution: Since $f(x)$ is odd we have $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=0$ and $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\ dx=0$ and $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\ dx=\frac{2}{\pi}\int_0^{\pi}(x^3-\pi^2x)\sin nx\ dx$. Integration by Parts gives

$$\int_0^{\pi}x\ \sin nx\ dx=\Big[-\tfrac{1}{n}\ x\ \cos nx\Big]_0^{\pi}+\int_0^{\pi}\tfrac{1}{n}\cos nx\ dx=-\tfrac{1}{n}\ \pi\ \cos n\pi=-\tfrac{(-1)^n\pi}{n}.$$

and

$$\begin{aligned}\int_0^{\pi}x^3\sin nx\ dx&=\Big[-\tfrac{1}{n}x^3\cos nx\Big]_0^{\pi}+\int_0^{\pi}\tfrac{3}{n}x^2\cos nx\ dx\\&=-\tfrac{(-1)^n\pi^3}{n}+\Big[\tfrac{3}{n^2}x^2\sin nx\Big]_0^{\pi}-\int_0^{\pi}\tfrac{6}{n^2}x\sin nx\ dx\\&=-\tfrac{(-1)^n\pi^3}{n}+0+\tfrac{6}{n^2}\,\tfrac{(-1)^n\pi}{n}=(-1)^n\Big(\tfrac{6\pi}{n^3}-\tfrac{\pi^3}{n}\Big)\end{aligned}$$

and so

$$b_n=\tfrac{2}{\pi}\int_0^{\pi}(x^3-\pi^2x)\sin nx\ dx=\tfrac{2}{\pi}\Big((-1)^n\big(\tfrac{6\pi}{n^3}-\tfrac{\pi^3}{n}\big)+(-1)^n\tfrac{\pi^3}{n}\Big)=\tfrac{(-1)^n12}{n^3}.$$

Since $f(x)=x^3-\pi^2x$ for $-\pi\le x\le\pi$, we have $f\big(\frac{\pi}{2}\big)=\big(\frac{\pi}{2}\big)^3-\pi^2\big(\frac{\pi}{2}\big)=-\frac{3\pi^3}{8}$. On the other hand, since $f(x)=\sum\limits_{n=1}^{\infty}\frac{(-1)^n12}{n^3}\sin nx$, and since when $n=2k$ we have $\sin\frac{n\pi}{2}=0$ and when $n=2k+1$ we have $\sin\frac{n\pi}{2}=(-1)^k$, we have $f\big(\frac{\pi}{2}\big)=\sum\limits_{n=1}^{\infty}\frac{(-1)^n12}{n^3}\sin\frac{n\pi}{2}=\sum\limits_{k=0}^{\infty}\frac{(-1)^{2k+1}12}{(2k+1)^3}(-1)^k=-\sum\limits_{k=0}^{\infty}\frac{(-1)^k12}{(2k+1)^3}$. Thus

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}=-\tfrac{1}{12}\,f\big(\tfrac{\pi}{2}\big)=\tfrac{1}{12}\cdot\tfrac{3\pi^3}{8}=\tfrac{\pi^3}{32}.$$

4: Use Fourier series to solve the ODE $4x'' + x = f(t)$, for $x = x(t)$, where $f : \mathbb{R} \to \mathbb{R}$ is the 2π-periodic function given by $f(t) = t^2$ for $-\pi \le t \le \pi$.

Solution: First let us find the Fourier series for $f(t)$. Since f is even we have $b_n = 0$ for all n, and we have We have

$$a_0 = \tfrac{1}{\pi}\int_0^\pi t^2\,dt = \tfrac{1}{\pi}\Big[\tfrac{1}{3}t^3\Big]_0^\pi = \tfrac{\pi^2}{3} \quad \text{, and}$$

$$\begin{aligned} a_n &= \tfrac{2}{\pi}\int_0^\pi t^2 \cos nt\;dt \\ &= \tfrac{2}{\pi}\left(\Big[\tfrac{1}{n}x^2\sin nx\Big]_0^\pi - \int_0^\pi \tfrac{2}{\pi}x\sin nx\;dx\right) \\ &= \tfrac{2}{\pi}\left(0 + \Big[\tfrac{2}{n^2}\,x\cos nx\Big]_0^\pi + \int_0^\pi \tfrac{2}{n^2}\cos nx\;dx\right) \\ &= \tfrac{2}{\pi}\Big(0 + \tfrac{2}{n^2}\pi(-1)^n + 0\Big) = \tfrac{4}{n^2}(-1)^n. \end{aligned}$$

Since $f(x)$ is equal to the sum of its Fourier series we have

$$f(x) = \tfrac{\pi^2}{3} + \sum_{n\ge 1} \tfrac{4(-1)^n}{n^2}\cos nt\,.$$

To solve the homogeneous DE $4x'' + x = 0$ we try $x = e^{rt}$: we need $4r^2 + 1 = 0$ so that $r = \pm\frac{1}{2}i$, and the general real solution is given by $x = x(t) = a\cos\frac{t}{2} + b\sin\frac{t}{2}$. By inspection, a particular solution to the DE $4x'' + x = \frac{\pi^2}{3}$ is given by the constant function $x = x_0(t) = \frac{\pi^2}{3}$. For each $n \in \mathbb{Z}^+$, to find a particular solution to the DE $4x'' + x = \cos nx$ we let $x = x_n(x) = A\cos nx + B\sin nx$ so that $x' = -An\sin nx + Bn\cos nx$ and $x'' = -An^2\cos nx - Bn^2\sin nx$. Put this in the DE to get

$$4\big(-An^2\cos nt - Bn^2\sin nt\big) + (A\cos nt + B\sin nt) = \cos nt$$

and equate the coefficients of $\cos nt$ and $\sin nt$ to get $-4An^2 + A = 1$ and $-4Bn^2 + B = 0$ so that $A = \frac{1}{1-4n^2}$ and $B = 0$. Thus a particular solution to the DE $4x'' + x = \cos nx$ is given by $x = x_n(t) = \frac{1}{1-4n^2}\cos nt$. Adding together multiples of the particular solutions $x = x_0(t)$ and $x = x_n(t)$, we find that a particular solution to the given DE $4x'' + x = f(x) = \frac{\pi^2}{3} + \sum\limits_{n\ge 1}\frac{4(-1)^n}{n^2}\cos nt$ is given by

$$x = x_p(t) = x_0(t) + \sum_{n\ge 1}\tfrac{4(-1)^n}{n^2}x_n(t) = \tfrac{\pi^2}{3} + \sum_{n\ge 1}\tfrac{4(-1)^n}{n^2(1-4n^2)}\cos nt.$$

Thus the general solution to the given ODE $4x'' + x = f(t)$ is given by

$$x(t) = a\cos\tfrac{t}{2} + b\sin\tfrac{t}{2} + \tfrac{\pi^2}{3} + \sum_{n\ge 1}\tfrac{4(-1)^n}{n^2(1-4n^2)}\cos nt.$$