**1:** The ODE $(1-x^2)y''-2xy'+k(k+1)y=0$ is called **Legendre's Equation**. For each integer $k\geq 0$, Legendre's equation has a unique polynomial solution $y=P_k(x)$ with $P_k(1)=1$. These are called the **Legendre polynomials**. Use power series, centred at 0, to solve the ODE, and find $P_k(x)$ for $k=0,1,2,3,4$.

**2:** The ODE $x^2y''+kxy'+\ell y=0$ is called the **Cauchy-Euler Equation**. We can solve the Cauchy-Euler equation by letting $y=x^r$ so that $y'=rx^{r-1}$ and $y''=r(r-1)x^{r-2}$. Putting these in the DE gives $0=r(r-1)x^r+krx^r+\ell x^r=\big(r(r-1)+kr+\ell\big)x^r$, so we see that $y=x^r$ is a solution when $r$ is a root of the polynomial $g(r)=r(r-1)+kr+\ell$.

(a) When $g(r)$ has two real roots $r_1$ and $r_2$, we obtain two independent solutions $y_1(x)=x^{r_1}$ and $y_2(x)=x^{r_2}$. Use this to solve the ODE $x^2y''-2xy+2y=0$.

(b) When $g(r)$ has complex roots $r\pm is$, we obtain the complex solutions $z_1(x)=x^{r+is}=e^{(r+is)\ln x}=e^{r\ln x}e^{is\ln x}=x^r\big(\cos(s\ln x)+i\sin(s\ln x)\big)$ and $z_2(x)=x^{r-is}=x^r\big(\cos(s\ln x)-i\sin(s\ln x)\big)$, and hence we obtain the two independent real solutions given by $y_1(x)=\frac{z_1(x)+z_2(x)}{2}=\mathrm{Re}(z_1(x))=x^r\cos(s\ln x)$ and $y_2(x)=\frac{z_1(x)-z_2(x)}{2i}=\mathrm{Im}(z_1(x))=x^r\sin(s\ln x)$. Use this to solve the ODE $x^2y''+3xy'+5y=0$.

(c) When $g(x)$ has a repeated real root $r$ we only obtain one solution $y_1(x)=x^r$. Use reduction of order to find a formula for a second independent solution $y=y_2(x)$ of the form $y_2(x)=y_1(x)u(x)$.

(d) Use the formula found in Part (c) to solve the ODE $x^2y''+5xy+4y=0$.

**3:** The ODE $x^2y''+xy'+(x^2-k^2)y=0$ is called **Bessel's Equation**. Use Frobenius' method to show that for all $k\geq 0$ there is a nonzero solution of the form $y=J_k(x)=x^k\sum_{n\geq 0}c_{2n}x^{2n}$ and, if $k$ is not an integer, there is a second independent solution of the form $y=J_{-k}(x)=x^{-k}\sum_{n\geq 0}c_{2n}x^{2n}$. These solutions (multiplied by various constants) are called the **Bessel functions of the first kind**.