SYDE Advanced Math 2, Solutions to Assignment 7

**1:** Find the $5^{\text{th}}$ Taylor polynomial, centred at 0, for the solution to the IVP $y'' + 2y' + e^x y = \sin x$ with $y(0) = 2$ and $y'(0) = 1$.

Solution: We try $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$. Then $y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \cdots$, $y'' = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots$, and

$$e^x y = \left(1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots\right)\left(a_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right)$$
$$= \left(a_0 + (c_1 + c_0)x + (c_2 + c_1 + \tfrac{1}{2}c_0)x^2 + (c_3 + c_2 + \tfrac{1}{2}c_1 + \tfrac{1}{6}c_0)x^3 + \cdots\right).$$

Put these in the DE to get

$$\left(2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots\right) + 2\left(c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots\right)$$
$$+ \left(c_0 + (c_1 + c_0)x + (c_2 + c_1 + \tfrac{1}{2}c_0)x^2 + (c_3 + c_2 + \tfrac{1}{2}c_1 + \tfrac{1}{6}c_0)x^3 + \cdots\right)$$
$$= \left(x - \tfrac{1}{6}x^3 + \cdots\right).$$

Equate coefficients to get $2c_2 + 2c_1 + c_0 = 0$ (1), $6c_3 + 4c_2 + c_1 + c_0 = 1$ (2), $12c_4 + 6c_3 + c_2 + c_1 + \tfrac{1}{2}c_0 = 0$ (3) and $20c_5 + 8c_4 + c_3 + c_2 + \tfrac{1}{2}c_1 + \tfrac{1}{6}c_0 = -\tfrac{1}{6}$ (4). To get $y(0) = 2$ we need $c_0 = 2$ and to get $y'(0) = 1$ we need $c_1 = 1$. Put these in the recursion formulas (1)-(4) to get $c_2 = \frac{-2c_1 - c_0}{2} = \frac{-2-2}{2} = -2$, $c_3 = \frac{1 - 4c_2 - c_1 - c_0}{6} = \frac{1+8-1-2}{6} = 1$, $c_4 = \frac{-6c_3 - c_2 - c_1 - \tfrac{1}{2}c_0}{12} = \frac{-6+2-1-1}{12} = -\tfrac{1}{2}$ and $c_5 = \frac{-\tfrac{1}{6} - 8c_4 - c_3 - c_2 - \tfrac{1}{2}c_1 - \tfrac{1}{6}c_0}{20} = \frac{-\tfrac{1}{6}+4-1+2-\tfrac{1}{2}-\tfrac{1}{3}}{20} = \tfrac{1}{5}$. Thus the $5^{\text{th}}$ Taylor polynomial is $T_5(x) = 2 + x - 2x^2 + x^3 - \tfrac{1}{2}x^4 + \tfrac{1}{5}x^5$.

**2:** Use the Power Series Method to solve the ODE $y'' + (x-1)y' + y = 0$. Find two linearly independent power series solutions, centred at 0, one satisfying the initial conditions $y(0) = 1$, $y'(0) = 0$, and the other satisfying $y(0) = 0$, $y'(0) = 1$. For each solution, state the recurrence relation for the coefficients, and find the $5^{\text{th}}$ Taylor polynomial centred at 0.

Solution: We try $y = \sum_{n \geq 0} c_n x^n$. Then $y' = \sum_{n \geq 1} n c_n x^{n-1}$ and $y'' = \sum_{n \geq 2}^{\infty} n(n-1)c_n x^{n-2}$. Put these in the DE to get

$$0 = y'' + (x-1)y' + y$$
$$= \sum_{n \geq 2} n(n-1)c_n x^{n-2} + \sum_{n \geq 1} n c_n x^n - \sum_{n \geq 1} n c_n x^{n-1} + \sum_{n \geq 0} c_n x^n$$
$$= \sum_{m \geq 0} (m+2)(m+1)c_{m+2} x^m + \sum_{m \geq 1} m c_m x^m - \sum_{m \geq 0} (m+1)c_{m+1} x^m + \sum_{m \geq 0} c_m x^m$$
$$= (2c_2 - c_1 + c_0)x^0 + \sum_{m \geq 1} \left((m+2)(m+1)c_{m+2} - (m+1)c_{m+1} + (m+1)c_m\right)x^m.$$

All coefficients vanish, so $c_2 = \frac{c_1 - c_0}{2}$ and $c_{m+2} = \frac{(m+1)c_{m+1} - (m+1)c_m}{(m+2)(m+1)} = \frac{c_{m+1} - c_m}{m+2}$ for $m \geq 1$. If $c_0 = 1$ and $c_1 = 0$ then the recursion formulas give $c_2 = \frac{0-1}{2} = -\tfrac{1}{2}$, $c_3 = \frac{-\tfrac{1}{2}-0}{3} = -\tfrac{1}{6}$, $c_4 = \frac{-\tfrac{1}{6}+\tfrac{1}{2}}{4} = \tfrac{1}{12}$ and $c_5 = \frac{\tfrac{1}{12}+\tfrac{1}{6}}{5} = \tfrac{1}{20}$, so the $5^{\text{th}}$ Taylor polynomial is

$$T_5(y_1) = 1 - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{12}x^4 + \tfrac{1}{20}x^5.$$

If $c_0 = 0$ and $c_1 = 1$ then the recursion formulas give $c_2 = \frac{1-0}{2} = \tfrac{1}{2}$, $c_3 = \frac{\tfrac{1}{2}-1}{3} = -\tfrac{1}{6}$, $c_4 = \frac{-\tfrac{1}{6}-\tfrac{1}{2}}{4} = -\tfrac{1}{6}$ and $c_5 = \frac{-\tfrac{1}{6}+\tfrac{1}{6}}{5} = 0$ and so the $5^{\text{th}}$ Taylor polynomial for the solution $y_2$ is

$$T_5(y_2) = x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 - \tfrac{1}{6}x^4.$$

**3:** Use Frobenius' Method to solve the ODE $4xy'' + 2y' = y$. Find two linearly independent series solutions, centred at 0. For each solution, solve the recurrence relation to obtain an explicit formula for the $n^{\text{th}}$ coefficient, then find a closed form formula for the solution.

Solution: We try $y = \sum\limits_{n\geq 0} c_n x^{n+r}$ so $y' = \sum\limits_{n\geq 0} (n+r)c_n x^{n+r-1}$ and $y'' = \sum\limits_{n\geq 0} (n+r)(n+r-1)c_n x^{n+r-2}$. Put these in the DE to get

$$0 = 4xy'' + 2y' - y$$

$$= \sum_{n\geq 0} 4(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n\geq 0} 2(n+r)c_n x^{n+r-1} - \sum_{n\geq 0} c_n x^{n+r}$$

$$= x^r \left( \sum_{m\geq -1} 4(m+r+1)(m+r)c_{m+1}x^m + \sum_{m\geq -1} 2(m+r+1)c_{m+1}x^m - \sum_{m\geq 0} c_m x^m \right)$$

$$= x^r \left( \sum_{m\geq -1} 2(m+r+1)(2m+2r+1)c_{m+1}x^m - \sum_{m\geq 0} c_m x^m \right)$$

$$= x^r \left( 2r(2r-1)c_0 x^{-1} + \sum_{m\geq 0} \left(2(m+r+1)(2m+2r+1)c_{m+1} - c_m\right)x^m \right).$$

All coefficients must vanish, so we have $r(2r-1) = 0$ and $c_{m+1} = \frac{c_m}{2(m+r+1)(2m+2r+1)}$ for $m \geq 0$. When $r = 0$, the recursion formula becomes $c_{m+1} = \frac{c_m}{2(m+1)(2m+1)} = \frac{c_m}{(2m+1)(2m+2)}$, so if $c_0 = 1$ then we get $c_1 = \frac{1}{1\cdot 2}$, $c_2 = \frac{1}{1\cdot 2\cdot 3\cdot 4}$, $c_3 = \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$, and in general $c_n = \frac{1}{(2n)!}$. In this case the solution is

$$y_1 = x^0 \left( \sum_{n=0}^{\infty} \frac{x^n}{(2n)!} \right) = \cosh \sqrt{x}\,.$$

When $r = \frac{1}{2}$ the recursion formula becomes $c_{m+1} = \frac{c_m}{2\left(m+\frac{3}{2}\right)(2m+2)} = \frac{c_m}{(2m+2)(2m+3)}$, so if $c_0 = 1$ then we get $c_1 = \frac{1}{2\cdot 3}$, $c_2 = \frac{1}{2\cdot 3\cdot 4\cdot 5}$ and in general $c_n = \frac{1}{(2n+1)!}$. In this case the solution is

$$y_2 = x^{1/2} \left( \sum_{n=0}^{\infty} \frac{x^n}{(2n+1)!} \right) = \sqrt{x} \cdot \frac{\sinh \sqrt{x}}{\sqrt{x}} = \sinh \sqrt{x}\,.$$

The general solution is $y = a \cosh \sqrt{x} + b \sinh \sqrt{x}$, for $x \geq 0$.

**4:** Use Frobenius' Method to solve the ODE $3x^2y'' + x(x-1)y' + y = 0$. Find two linearly independent series solutions, centred at 0. For each solution, solve the recurrence relation to obtain an explicit formula for the $n^{\text{th}}$ coefficient. Find a closed form formula for one of the two solutions.

Solution: We try $y = \sum_{n \geq 0} c_n x^{n+r}$ so $y' = \sum_{n \geq 0} (n+r)c_n x^{n+r-1}$ and $y'' = \sum_{n \geq 0} (n+r)(n+r-1)c_n x^{n+r-2}$. Put these in the DE to get

$$0 = 3x^2 y'' + x(x-1)y' + y$$

$$= \sum_{n \geq 0} 3(n+r)(n+r-1)c_n x^{n+r} + \sum_{n \geq 0} (n+r)c_n x^{n+r+1} - \sum_{n \geq 0} (n+r)c_n x^{n+r} + \sum_{n \geq 0} c_n x^{n+r}$$

$$= x^r \left( \sum_{m \geq 0} 3(m+r)(m+r-1)c_m x^m + \sum_{m \geq 1} (m+r-1)c_{m-1}x^m - \sum_{m \geq 0} (m+r)c_m x^m + \sum_{m \geq 0} a_m x^m \right)$$

$$= x^r \left( \sum_{m \geq 0} \left(3(m+r)(m+r-1) - (m+r) + 1\right)c_m x^m + \sum_{m \geq 1} (m+r-1)c_{m-1}x^m \right)$$

$$= x^r \left( \left(3r(r-1) - r + 1\right)c_0 x^0 + \sum_{m \geq 1} \left(\left(3(m+r)(m+r-1) - (m+r) + 1\right)c_m + (m+r-1)c_{m-1}\right)x^m \right).$$

All coefficients must vanish, so we have $3r(r-1) - r + 1 = 0$, that is $3r^2 - 4r + 1 = 0$ or equivalently $(3r - 1)(r - 1) = 0$ so $r = 1$ or $r = \frac{1}{3}$, and we have $c_m = \frac{-(m+r-1)c_{m-1}}{3(m+r)(m+r-1)-(m+r)+1} = \frac{-c_{m-1}}{3(m+r)-1}$.

When $r = 1$ the recursion formula becomes $c_m = \frac{-c_{m-1}}{3m+2}$. If we take $c_0 = 1$ then we have $c_1 = -\frac{1}{5}$, $c_2 = \frac{1}{5\cdot 8}$, $c_3 = -\frac{1}{5\cdot 8\cdot 11}$, and in general $c_n = \frac{(-1)^n}{5\cdot 8\cdot 11\cdots (3n+2)}$. In this case we obtain the solution

$$y_1 = x^1 \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{5\cdot 8\cdot 11\cdots (3n+2)} \right) = x - \frac{1}{5}x^2 + \frac{1}{5\cdot 8}x^3 - \frac{1}{5\cdot 8\cdot 11}x^4 + \cdots.$$

When $r = \frac{1}{3}$ the recursion formula becomes $c_m = -\frac{c_{m-1}}{3m}$. If we set $c_0 = 1$ then we obtain $c_2 = -\frac{1}{3}$, $c_2 = \frac{1}{3\cdot 6}$, $c_3 = -\frac{1}{3\cdot 6\cdot 9}$, and in general $c_n = \frac{(-1)^n}{3\cdot 6\cdot 9\cdots (3n)} = \frac{(-1)^n}{3^n n!}$. In this case we obtain the solution

$$y_2 = x^{1/3} \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n n!} \right) = x^{1/3} e^{-x/3}.$$