SYDE Advanced Math 2, Solutions to Assignment 11

1: The following ODEs are examples of Sturm-Liouville boundary value problems involving a parameter k. In each case, non-zero solutions only occur for certain values of k, which are called *eigenvalues* and the corresponding solutions are called *eigenfunctions*.

(a) Find the possible values of $k \in \mathbb{R}$ and the non-zero solutions the ODE $u'' = ku$ for $u = u(x)$ satisfying the boundary conditions $u'(0) = 0$ and $u(1) = 0$.

Solution: When $k = 0$, the DE $u'' = ku$ becomes $u'' = 0$, which has solutions $u(x) = ax + b$ with $u'(x) = a$. To get $u'(0) = 0$ we need $a = 0$ so that $u(x) = b$, and then to get $u(1) = 0$ we need $b = 0$, so we only obtain the zero solution. Suppose $k > 0$ with $k = \sigma^2$, the DE becomes $u'' - \sigma^2 u = 0$, which has solutions $u(x) = ae^{\sigma x} + be^{-\sigma x}$ with $u'(x) = \sigma a e^{\sigma x} - \sigma b e^{\sigma x}$. To get $\sigma'(0) = 0$ we need $\sigma a - \sigma b = 0$, that is $\sigma a = \sigma b$, and hence $a = b$ (since $\sigma > 0$). To get $u(1) = 0$ we need $0 = ae^{\sigma} + be^{-\sigma} = ae^{\sigma} + ae^{-\sigma} = a(e^{\sigma} + e^{-\sigma})$ and hence $a = 0$ (since $e^{\sigma} + e^{-\sigma} > 0$). Thus when $k > 0$ we only obtain the zero solution. Suppose that $k < 0$, say $k = -\sigma^2$ with $\sigma > 0$. The DE becomes $u'' + \sigma^2 u = 0$ which has solutions $u(x) = a \sin \sigma x + b \cos \sigma x$ with $u'(x) = \sigma a \cos \sigma x - \sigma b \sin \sigma x$. To get $u'(0) = 0$ we need $a = 0$ so that $u(x) = b \cos \sigma x$. To get $u(1) = 0$ we need $b \cos \sigma = 0$. When $b = 0$ we obtain the zero-solution, so for a non-zero solution we need $\cos \sigma = 0$ which occurs when $\sigma = \frac{\pi}{2} + n\pi$ for some $0 \leq n \in \mathbb{Z}$. Thus the values of k for which a non-zero solution exists are the values $k = -\sigma^2 = -(\frac{\pi}{2} + n\pi)^2$ with $0 \leq n \in \mathbb{Z}$, and the corresponding solutions are given by $u(x) = u_n(x) = b_n \cos \sigma x = b_n \cos\big((\frac{\pi}{2} + n\pi)x\big)$.

(b) Find the possible values of $k \in \mathbb{R}$ and the non-zero solutions to the ODE $x^2 u'' + xu' + ku = 0$ satisfying the boundary conditions $u(1) = 0$ and $u(4) = 0$.

Solution: This DE is a Cauchy-Euler equation. We found the solutions to Cauchy-Euler equations in Question 2 on Problem Set 4. To solve the DE $x^2 u'' + xu' + ku = 0$, we let $y = x^r$ and put this in the DE to get $r(r-1) + r + k = 0$, that is $r^2 + k = 0$. When $k = 0$ we find that $r = 0$ (a repeated real root) so the solutions to the DE are given by $u(x) = a + b \ln x$. To get $u(1) = 0$ we need $a = 0$ so that $u(x) = b \ln x$, then to get $u(4) = 0$ we need $b = 0$ giving the zero solution. When $k < 0$, say $k = -\sigma^2$ with $\sigma > 0$, the equation $r^2 + k = 0$ becomes $r^2 - \sigma^2 = 0$ giving $r = \pm\sigma$, so the solutions to the DE are given by $u(x) = ax^{\sigma} + bx^{-\sigma}$. To get $u(1) = 0$ we need $a + b = 0$ so $b = -a$ and $u(x) = a(x^{\sigma} - x^{-\sigma})$. To get $u(4) = 0$ we need $a(4^{\sigma} - 4^{-\sigma}) = 0$ and hence $a = 0$ (because $4^{\sigma} > 1$ and $4^{-\sigma} < 1$ so that $4^{\sigma} - 4^{-\sigma} > 0$), so we only obtain the zero solution. Suppose that $k > 0$, say $k = \sigma^2$.The equation $r^2 + k = 0$ becomes $r^2 + \sigma^2 = 0$ so that $r = \pm\sigma i$, and the solutions to the DE are given by $u(x) = a\cos(\sigma \ln x) + b \sin(\sigma \ln x)$. To get $u(1) = 0$ we need $a = 0$ so that $u(x) = b \sin(\sigma \ln x)$. To get $u(4) = 0$, for a non-zero solution we need $\sin(\sigma \ln 4) = 0$, so we must have $\sigma \ln 4 = n\pi$ for some positive integer n. Thus the values of k for which there exists a non-zero solution are $k = \sigma^2 = (\frac{n\pi}{\ln 4})^2$ with $0 < n \in \mathbb{Z}$, and the corresponding solutions are $u(x) = u_n(x) = b_n \sin\big(\frac{n\pi}{\ln 4} \ln x\big)$.

2: Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x}^2$ for $u = u(x,t)$ with $0 \le x \le 4$ and $t \ge 0$, satisfying the fixed endpoint condition $u(0,t) = u(4,t) = 0$ for all $t \ge 0$ and the initial conditions $u(x,0) = 0$ and $\frac{\partial u}{\partial t}(x,0) = 2\sin\frac{\pi x}{4}$ for $0 \le x \le 4$.

Solution: We know (from Example 4.10 in the Lecture Notes) that the solution to the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ with $u(0,t) = u(\ell,t) = 0$ for $t \ge 0$ and $u(x,0) = 0$ and $\frac{\partial u}{\partial t}(x,0) = g(t)$ for $0 \le x \le \ell$ is given by $u(x,t) = \sum\limits_{n=1}^{\infty} d_n \sin\left(\frac{cn\pi}{\ell}t\right)\sin\left(\frac{n\pi}{\ell}x\right)$ where the constants $\frac{cn\pi}{\ell}d_n$ are the Fourier coefficients for the odd 2ℓ-periodic function which is equal to $g(x)$ for $0 \le x \le \ell$. In this problem, we take $c = 2$ and $\ell = 4$ and $g(x) = 2\sin\left(\frac{\pi}{4}x\right)$. Note that $g(x)$ is already in the form of a trigonometric polynomial for an 8-periodic function, and we have $g(x) = \sum\limits_{n=1}^{\infty} b_n \sin(\frac{n\pi}{4}x)$ with Fourier coefficients $b_1 = 2$ and $b_n = 0$ for $n \ge 2$. To get $\frac{cn\pi}{\ell}d_n = b_n$ with $c = 2$ and $\ell = 4$, we need $d_n = \frac{2}{n\pi}b_n$, so $d_1 = \frac{4}{\pi}$ and $d_n = 0$ for $n \ge 2$. The solution is

$$u(x,t) = \sum_{n=1}^{\infty} d_n \sin\left(\tfrac{cn\pi}{\ell}t\right)\sin\left(\tfrac{n\pi}{\ell}x\right) = \tfrac{4}{\pi}\sin\left(\tfrac{\pi}{2}t\right)\sin\left(\tfrac{\pi}{4}x\right).$$

3: Solve the heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ for $u = u(x,t)$ with $0 \le x \le \ell$ and $t \ge 0$ satisfying the fixed endpoint temperature condition $u(0,t) = 0$ and $u(\ell,t) = 0$ for all $t \ge 0$ and the initial condition $u(x,0) = f(x)$ for all $0 \le x \le \ell$ where $f(x)$ is given by $f(x) = 0$ for $0 \le x < \frac{1}{4}\ell$, $f(x) = 1$ for $\frac{1}{4} < x < \frac{3\ell}{4}$ and $f(x) = 0$ for $\frac{3\ell}{4} < x \le \ell$ (with $f\left(\frac{\ell}{4}\right) = f\left(\frac{3\ell}{4}\right) = \frac{1}{2}$ so that $f(x)$ is equal to the sum of its Fourier series).

Solution: We know (from Exercise 4.13 in the Lecture Notes) that the solution to the heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ satisfying the fixed endpoint temperature conditions $u(0,t) = u(\ell,t) = 0$ for $t \ge 0$ and the initial condition $u(x,0) = f(x)$ for $0 \le x \le \ell$ is given by $u(x,t) = \sum_{n=0}^{\infty} b_n e^{-(cn\pi/\ell)^2 t}\cos\left(\frac{n\pi}{\ell}x\right)$ where the b_n are the Fourier coefficients of the odd 2ℓ-periodic function which is equal to $f(x)$ for $0 \le x \le \ell$. We need

$$b_n = \tfrac{2}{\ell}\int_0^\ell f(x)\sin\left(\tfrac{n\pi}{\ell}x\right)dx = \tfrac{2}{\ell}\int_{\ell/4}^{3\ell/4}\sin\left(\tfrac{n\pi}{\ell}x\right)x = \tfrac{2}{\ell}\Big[-\left(\tfrac{\ell}{n\pi}\right)\cos\left(\tfrac{n\pi}{\ell}x\right)\Big]_{\ell/4}^{3\ell/4} = \tfrac{2}{n\pi}\left(\cos\tfrac{n\pi}{4} - \cos\tfrac{3n\pi}{4}\right).$$

We have two sequences, both of period 8, given by $\left(\cos\frac{n\pi}{4}\right)_{n\ge 0} = \left(1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 1, \cdots\right)$ and $\left(\cos\frac{3n\pi}{4}\right)_{n\ge 0} = \left(1, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, -1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, 1, \cdots\right)$, and subtracting the second from the first gives $\left(\cos\frac{n\pi}{4} - \cos\frac{3n\pi}{4}\right)_{n\ge 0} = \left(0, \sqrt{2}, 0, -\sqrt{2}, 0, -\sqrt{2}, 0, \sqrt{2}, 0, \cdots\right)$. Thus the coefficients are given by $b_n = 0$ when n is even, and $b_n = \frac{2\sqrt{2}}{n\pi}$ when $n = \pm 1 + 8k$, and $b_n = -\frac{2\sqrt{2}}{n\pi}$ when $n = \pm 3 + 8k$, which we can write as $b_n = (-1)^{(n-1)(n-7)/8}$ when n is odd. The solution is

$$u(x,t) = \sum_{n \text{ odd}} (-1)^{\frac{(n-1)(n-7)}{8}}\,\tfrac{2\sqrt{2}}{n\pi}\,e^{(cn\pi/\ell)^2 t}\cos\left(\tfrac{n\pi}{\ell}x\right).$$

4: Solve the heat equation $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$ for $u=u(x,t)$ with $0\leq x\leq \ell$ and $t\geq 0$ satisfying the insulated ends condition $\frac{\partial u}{\partial x}(0,t)=0$ and $\frac{\partial u}{\partial x}(\ell,t)=0$ for all $t\geq 0$ and the initial condition $u(x,0)=f(x)$ for all $0\leq x\leq \ell$ where $f(x)$ is given by $f(x)=1$ for $0<x<\frac{2\ell}{3}$ and $f(x)=3$ for $\frac{2\ell}{3}<x<\ell$ (with $f(0)=f(\frac{2\ell}{3})=f(1)=2$).

Solution: We know (from Exercise 4.14) that the solution is given by $u(x,t)=\sum_{n=0}^\infty d_n e^{-(cn\pi/\ell)^2 t}\cos\left(\frac{n\pi}{\ell}x\right)$ where the a_n are the Fourier coefficients of the even 2ℓ-periodic function which is equal to $f(x)$ for $0\leq x\leq \ell$. We need

$$\begin{aligned}
a_0&=\tfrac{1}{\ell}\int_{x=0}^{\ell} f(x)\,dx=\tfrac{1}{\ell}\left(\int_0^{2\ell/3}1\,dx+\int_{2\ell/3}^{\ell}3\,dx\right)=\tfrac{1}{\ell}\left(\tfrac{2\ell}{3}+\ell\right)=\tfrac{5}{3}\\
a_n&=\tfrac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\tfrac{n\pi}{\ell}x\right)dx=\tfrac{2}{\ell}\left(\int_0^{2\ell/3}\cos\left(\tfrac{n\pi}{\ell}x\right)dx+\int_{2\ell/3}^{\ell}3\cos\left(\tfrac{n\pi}{\ell}x\right)dx\right)\\
&=\tfrac{2}{\ell}\left(\left[\tfrac{\ell}{n\pi}\sin\left(\tfrac{n\pi}{\ell}x\right)\right]_0^{2\ell/3}+\left[\tfrac{3\ell}{n\pi}\sin\left(\tfrac{n\pi}{\ell}x\right)\right]_{2\ell/3}^{\ell}\right)=\tfrac{2}{\ell}\left(\tfrac{\ell}{n\pi}\sin\tfrac{2n\pi}{3}-\tfrac{3\ell}{n\pi}\sin\tfrac{2n\pi}{3}\right)\\
&=-\tfrac{4}{n\pi}\sin\tfrac{2n\pi}{3}=\left\{\begin{array}{cl}0 & \text{if } n=0+3k\\ -\frac{2\sqrt{3}}{n\pi} & \text{if } n=1+3k\\ \frac{2\sqrt{3}}{n\pi} & \text{if } n=2+3k\end{array}\right\}.
\end{aligned}$$

The solution is

$$\begin{aligned}
u(x,t)&=\tfrac{5}{3}-\sum_{n=0}^{\infty}\tfrac{4}{n\pi}\sin\tfrac{2n\pi}{3}\,e^{(cn\pi/\ell)^2 t}\cos\left(\tfrac{n\pi}{\ell}x\right)\\
&=\tfrac{5}{3}-\sum_{n=1+3k}\tfrac{2\sqrt{3}}{n\pi}\,e^{(cn\pi/\ell)^2 t}\cos\left(\tfrac{n\pi}{\ell}x\right)+\sum_{n=2+3k}\tfrac{2\sqrt{3}}{n\pi}\,e^{(cn\pi/\ell)^2 t}\cos\left(\tfrac{n\pi}{\ell}x\right).
\end{aligned}$$

5: Solve Dirichlet's problem, that is solve Laplace's equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, for $u=u(x,y)$ on the square $0\leq x\leq 1$, $0\leq y\leq 1$ satisfying the boundary conditions $u(x,0)=x$ and $u(x,1)=x$ for $0\leq x\leq 1$, and $u(0,y)=\sin\pi y$ and $u(1,y)=1-\sin\pi y$ for $0\leq y\leq 1$.

Solution: Note that $v=v(x,y)=x$ satisfies Laplace's equation with $v(x,0)=v(1,0)=x$ and $v(0,y)=0$ and $v(1,y)=1$. If $u=u(x,y)$ is the desired solution and $w=u-v$, then we will have $w(x,0)=0$, $w(x,1)=0$, $w(0,y)=\sin\pi y$ and $w(1,y)=-\sin\pi y$. Following the method of Example 4.15, we find two functions $w=w_3(x,y)$ and $w=w_4(x,y)$ where $w_3(0,y)=f_3(y)=\sin\pi y$ and is zero on the other 3 edges of the square, and $w_4(1,y)=f_4(y)=-\sin\pi y$ and iz zero on the other 3 edges of the square. As shown in Example 4.15, the function $w_3(x,y)$ is given by $w_3(x,y)=\sum_{n=1}^{\infty}c_n\sinh(n\pi(1-x))\sin(n\pi y)$ where the constants $c_n\sinh(n\pi)$ are the Fourier coefficients of the odd 2-periodic function which is equal to $f_3(y)=\sin\pi y$. Note that $f_3(y)$ is already in the form of a trigonometric polynomial, so we see that its Fourier coefficients are given by $b_1=1$ and $b_n=0$ for $n\neq 1$, so we have $c_1=\frac{1}{\sinh\pi}$ and $c_n=0$ for $n\neq 1$. Thus

$$w_3(x,y)=\tfrac{1}{\sinh\pi}\sinh(\pi(1-x))\sin(\pi y).$$

Using a similar (but slightly easier) argument to the argument used in Example 4.15, or (more easily) by using symmetry (by replacing $1-x$ by x and noting that $f_4(y)=-f_3(y)$) we see that

$$w_4(x,y)=-\tfrac{1}{\sinh\pi}\sinh(\pi x)\sin(\pi y).$$

Thus the solution $u=u(x,y)$ to the given problem is

$$u(x,y)=v(x,y)+w_3(x,y)+w_4(x,y)=x+\tfrac{1}{\sinh\pi}\big(\sinh(\pi(1-x))-\sinh(\pi x)\big)\sin(\pi y).$$

6: Consider Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(a) Change to polar coordinates by letting $x = r\cos\theta$ and $y = r\sin\theta$. Use the Chain Rule to calculate $\frac{\partial u}{\partial r}$ and $\frac{\partial^2 u}{\partial r^2}$, and $\frac{\partial u}{\partial \theta}$ and $\frac{\partial^2 u}{\partial \theta^2}$, and hence show that Laplace's equation, for $u = u(r,\theta) = u(x(r,\theta), y(r,\theta))$, becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Solution: By the Chain Rule, we have

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
\frac{\partial^2 u}{\partial r^2} &= \left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial r}\right)\cos\theta + \left(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r}\right)\sin\theta \\
&= \left(\frac{\partial^2 u}{\partial x^2}\cos\theta + \frac{\partial^2 u}{\partial x \partial y}\sin\theta\right)\cos\theta + \left(\frac{\partial^2 u}{\partial x \partial y}\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin\theta\right)\sin\theta \\
&= \cos^2\theta\,\frac{\partial^2 u}{\partial x^2} + \sin^2\theta\,\frac{\partial^2 u}{\partial y^2} + 2\sin\theta\cos\theta\,\frac{\partial^2 u}{\partial x \partial y}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial u}{\partial x}\,r\sin\theta + \frac{\partial u}{\partial y}\,r\cos\theta \\
\frac{\partial^2 u}{\partial \theta^2} &= \left(-\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \theta} - \frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial \theta}\right)r\sin\theta - \frac{\partial u}{\partial x}\,r\cos\theta + \left(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \theta}\right)r\cos\theta - \frac{\partial u}{\partial y}\,r\sin\theta \\
&= \left(\frac{\partial^2 u}{\partial x^2}\,r\sin\theta - \frac{\partial^2 u}{\partial x \partial y}\,r\cos\theta\right)r\sin\theta - \frac{\partial u}{\partial x}\,r\cos\theta + \left(-\frac{\partial^2 u}{\partial x \partial y}\,r\sin\theta + \frac{\partial^2 u}{\partial y^2}\,r\cos\theta\right)r\cos\theta - \frac{\partial u}{\partial y}\,r\sin\theta \\
&= r^2\sin^2\theta\,\frac{\partial^2 u}{\partial x^2} + r^2\cos^2\theta\,\frac{\partial^2 u}{\partial x \partial y} - 2r\sin\theta\cos\theta\,\frac{\partial^2 u}{\partial x \partial y} - r\cos\theta\,\frac{\partial u}{\partial x} - r\sin\theta\,\frac{\partial u}{\partial y}
\end{aligned}$$

so that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

(b) Find a solution $u = u(x,y)$ to Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in the annulus given by $1 \le x^2 + y^2 \le 2$ satisfying the boundary conditions $u(x,y) = 6$ when $x^2 + y^2 = 1$ and $u(x,y) = 10$ when $x^2 + y^2 = 2$.

Solution: By symmetry, we look for a solution of the form $u = u(r)$ to Laplace's equation in polar coordinates with $u(1) = 6$ and $u(\sqrt{2}) = 10$. When $u = u(r)$, Laplace's equation in polar coordinates becomes $u'' + \frac{1}{r}u' = 0$. Letting $v = v(r) = u'(r)$ and $v'(r) = u''(r)$, the DE becomes $v' + \frac{1}{r}v = 0$, which is linear. An integrating factor is $\lambda = e^{\int \frac{1}{r}\,dr} = e^{\ln r} = r$, and the solution is given by $v(r) = \frac{1}{r}\int 0\,dr = \frac{a}{r}$, that is $u' = \frac{a}{r}$. Integrate to get $u = a\ln r + b$. To get $u(1) = 6$ we need $b = 6$ so that $u(r) = 6 + a\ln x$. Then to get $u(\sqrt{2}) = 10$ we need $6 + a\ln\sqrt{2} = 10$, so we must take $a = \frac{4}{\ln\sqrt{2}} = \frac{8}{\ln 2}$ and the solution is $u(r) = 6 + \frac{8}{\ln 2}\ln r$. In Cartesian coordinates, this can be written as $u(x,y) = 6 + \frac{8}{\ln 2}\ln\sqrt{x^2+y^2} = 6 + \frac{4}{\ln 2}\ln(x^2+y^2) = 6 + 4\log_2(x^2+y^2)$.