

# SYDE Advanced Math 2, Solutions to Assignment 10

- 1: (a) Use the method of separation of variables to find a solution  $u = u(x, y)$  to the PDE  $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x + y)u$  with  $u(1, 0) = 4$  and  $u(0, 1) = 1$ .

Solution: Let  $u(x, y) = X(x)Y(y)$ . Then  $\frac{\partial u}{\partial x} = X'Y$  and  $\frac{\partial u}{\partial y} = XY'$ . Putting this in the PDE gives  $X'Y + XY' = 2(x + y)XY$ . Dividing by  $XY$  gives  $\frac{X'}{X} + \frac{Y'}{Y} = 2x + 2y$ , that is  $\frac{X'(x)}{X(x)} - 2x = 2y - \frac{Y'(y)}{Y(y)}$ . Since the left side depends only on  $x$  and the right side depends only on  $y$ , in order to be equal for all  $x$  and  $y$  both sides must be constant, say  $\frac{X'}{X} - 2x = k = 2y - \frac{Y'}{Y}$ . The DE  $\frac{X'}{X} - 2x = k$  is linear, since we can write it as  $X' - (2x + k)X = 0$ , an integrating factor is given by  $\lambda = e^{-x^2 - kx}$  and the solution is given by  $X = a e^{x^2 + kx}$ . Similarly, the DE  $k = 2y - \frac{Y'}{Y}$  is linear as it can be written as  $Y' - (2y - k)Y = 0$  and the solution is  $Y = b e^{y^2 - y}$ . Thus we obtain the solution  $u(x, y) = X(x)Y(y) = c e^{x^2 + kx} e^{y^2 - ky} = c e^{x^2 + y^2 + k(x - y)}$  to the original PDE (where  $c = ab$ ). To get  $u(1, 0) = 4$  we need  $ce^{1+k} = 4$ , that is  $ce^k = 4$  (1) and to get  $u(0, 1) = 1$  we need  $ce^{1-k} = 1$ , that is  $ce^{-k} = 1$  (2). Dividing equation (1) by equation (2) gives  $e^{2k} = 4$  so that  $k = \ln 2$ , and putting this in equation (1) gives  $c = \frac{2}{e}$ . Thus we obtain the solution

$$u = c e^{x^2 + y^2 + k(x - y)} = \frac{2}{e} e^{x^2 + y^2 + (x - y) \ln 2} = \frac{2}{e} e^{x^2 + y^2} 2^{x - y} = 2^{x - y + 1} e^{x^2 + y^2 - 1}.$$

- (b) Solve the PDE given by  $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = y$  for  $u = u(x, y)$  with  $u(x, y) = 1$  on the line  $x + y = 1$  by making a change of variables, letting  $r = x$  and  $s = y - 2x$ .

Solution: Let  $r = x$  and  $s = y - 2x$  and note that  $x = r$  and  $y = s + 2x = s + 2r$ . By the Chain Rule, we have

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} = \frac{\partial u}{\partial r} - 2 \frac{\partial u}{\partial s} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial y} = \frac{\partial u}{\partial s} \\ \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial y} &= \left( \frac{\partial u}{\partial r} - 2 \frac{\partial u}{\partial s} \right) + 2 \left( \frac{\partial u}{\partial s} \right) = \frac{\partial u}{\partial r},\end{aligned}$$

so the PDE becomes  $\frac{\partial u}{\partial r} = s + 2r$ . This gives  $u = \int s + 2r \, dr = sr + r^2 + k(s)$ . The line  $x + y = 1$  in the new coordinates  $r, s$  becomes  $r + (s + 2r) = 1$ , that is  $3r + s = 1$ , so to have  $u = 1$  on the line  $x + y = 1$  we need  $1 = sr + r^2 + k(s)$  whenever  $r = \frac{1-s}{3}$ , which implies  $k(s) = 1 - sr - r^2 = 1 - s\left(\frac{1-s}{3}\right) - \left(\frac{1-s}{3}\right)^2 = \frac{1}{9}(8 - s + s^2)$ . Thus the solution is given by

$$\begin{aligned}u &= sr + r^2 + k(s) = sr + r^2 + \frac{1}{9}(8 - s + s^2) \\ &= \frac{1}{9}(9sr + 9r^2 + 8 - s + 2s^2) \\ &= \frac{1}{9}(9(y - 2x)x + 9x^2 + 8 - (y - 2x) + 2(y - 2x)^2) \\ &= \frac{1}{9}(-x^2 + xy + 2y^2 + 2x - y + 8).\end{aligned}$$

- 2: (a) Use separation of variables and Fourier series to solve the wave equation  $\frac{\partial^2 u}{\partial t^2} = \frac{1}{4} \frac{\partial^2 u}{\partial x^2}$  for  $u = u(x, t)$  with  $0 \leq x \leq 2$  and  $t \in \mathbb{R}$  satisfying the fixed ends condition  $u(0, t) = u(2, t) = 0$  for all  $t \in \mathbb{R}$  and the initial conditions  $u(x, 0) = (\sin \pi x)(1 + \cos \pi x)$  and  $\frac{\partial u}{\partial t}(x, 0) = 0$  for all  $0 \leq x \leq 2$ .

Solution: Let  $f(x) = (\sin \pi x)(1 + \cos \pi x) = \sin \pi x + \sin \pi x \cos \pi x = \sin \pi x + \frac{1}{2} \sin 2\pi x$ . We know (see Example 4.10) that the solution  $u = u(x, t)$  to the wave equation with  $c = \frac{1}{2}$  and  $\ell = 2$  with fixed ends satisfying  $u(x, 0) = f(x)$  and  $\frac{\partial u}{\partial t}(x, 0) = 0$  is given by

$$u(x, t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{cn\pi}{\ell}t\right) \sin\left(\frac{n\pi}{\ell}x\right) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{4}t\right) \sin\left(\frac{n\pi}{2}x\right)$$

where the  $a_n$  are the Fourier coefficients of the odd function of period  $2\ell = 4$  equal to  $f(x) = \sin \pi x + \frac{1}{2} \sin 2\pi x$  for  $0 \leq x \leq 2$ . By inspection, the Fourier coefficients of  $f(x)$  are given by  $a_2 = 1$  and  $a_4 = \frac{1}{2}$  and  $a_n = 0$  for all  $n \neq 2, 4$ , so the solution is

$$u(x, t) = \cos\left(\frac{\pi}{2}t\right) \sin(\pi x) + \frac{1}{2} \cos(\pi t) \sin(2\pi x).$$

- (b) Find a constant  $c$  and function  $g(x)$  such that  $u(x, t) = g(x + ct) + g(x - ct)$  for all  $x, t$  (and show that this is the case).

Solution: Let  $c = \frac{1}{2}$  and  $g(x) = \frac{1}{2}f(x) = \frac{1}{2} \sin(\pi x) + \frac{1}{4} \sin(2\pi x)$ . Then

$$\begin{aligned} g(x + ct) &= g\left(x + \frac{t}{2}\right) = \frac{1}{2} \sin\left(\pi\left(x + \frac{t}{2}\right)\right) + \frac{1}{4} \sin\left(2\pi\left(x + \frac{t}{2}\right)\right) \\ &= \frac{1}{2} \sin\left(\pi x + \frac{\pi t}{2}\right) + \frac{1}{4} \sin\left(2\pi x + \pi t\right) \\ &= \frac{1}{2} \left( \sin(\pi x) \cos\left(\frac{\pi t}{2}\right) + \cos(\pi x) \sin\left(\frac{\pi t}{2}\right) \right) + \frac{1}{4} \left( \sin(2\pi x) \cos(\pi t) + \cos(2\pi x) \sin(\pi t) \right) \end{aligned}$$

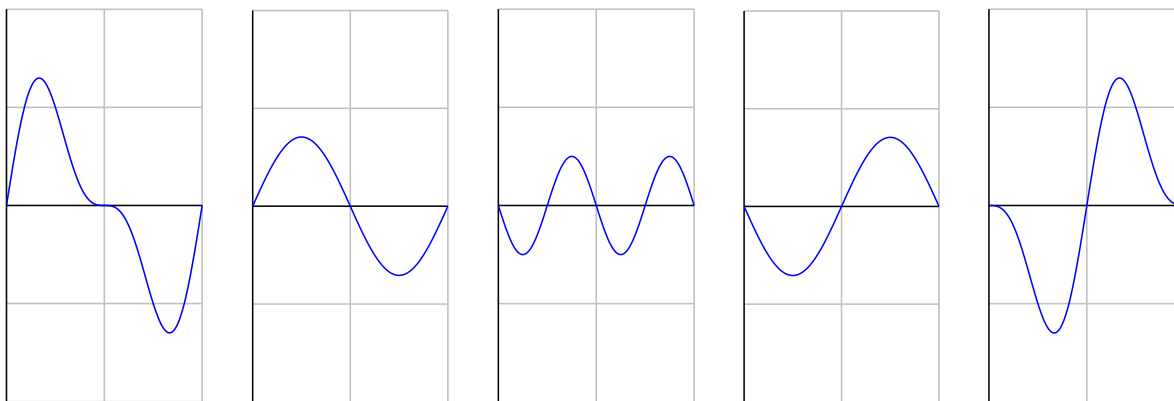
and similarly

$$\begin{aligned} g(x - ct) &= g\left(x - \frac{t}{2}\right) = \frac{1}{2} \sin\left(\pi x - \frac{\pi t}{2}\right) + \frac{1}{4} \sin\left(2\pi x - \pi t\right) \\ &= \frac{1}{2} \left( \sin(\pi x) \cos\left(\frac{\pi t}{2}\right) - \cos(\pi x) \sin\left(\frac{\pi t}{2}\right) \right) + \frac{1}{4} \left( \sin(2\pi x) \cos(\pi t) - \cos(2\pi x) \sin(\pi t) \right) \end{aligned}$$

and so  $g(x + ct) + g(x - ct) = \sin(\pi x) \cos\left(\frac{\pi t}{2}\right) + \frac{1}{2} \sin(2\pi x) \cos(\pi t) = u(x, t)$ .

- (c) By plotting points, accurately sketch the graphs  $u = u(x, t)$  (in the  $xu$ -plane) for  $t = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ .

Solution: We have  $u(x, 0) = \sin(\pi x) + \frac{1}{2} \sin(2\pi x)$ , and  $u(x, \frac{1}{2}) = \frac{\sqrt{2}}{2} \sin(\pi x)$ , and  $u(x, 1) = -\frac{1}{2} \sin(2\pi x)$ , and  $u(x, \frac{3}{2}) = -\frac{\sqrt{2}}{2} \sin(\pi x)$ , and  $u(x, 2) = -\sin(\pi x) + \frac{1}{2} \sin(2\pi x)$ . The graphs are shown below:



- 3: (a) Solve the heat equation  $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$  for  $u = u(x, t)$  with  $0 \leq x \leq \pi$  and  $t \geq 0$  satisfying the insulated ends condition  $\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0$  for all  $t \geq 0$  and the initial condition  $u(x, 0) = x^2$  for all  $0 \leq x \leq \pi$ .

Solution: We know (Exercise 4.14) that the solution to the heat equation with  $c = 1$  and  $\ell = \pi$  with insulated ends satisfying  $u(x, 0) = f(x) = x^2$  is given by

$$u(x, t) = \sum_{n=0}^{\infty} a_n e^{-n^2 t} \cos nx$$

where the constants  $a_n$  are the Fourier coefficients of the even  $2\pi$ -periodic function which is equal to  $f(x) = x^2$  for  $0 \leq x \leq \pi$ . The coefficients are

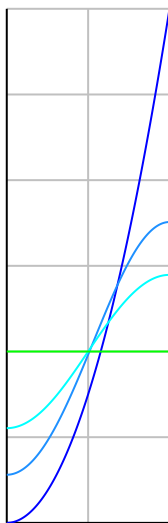
$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{\pi} x^2 dx = \frac{1}{\pi} \left[ \frac{1}{3} x^3 \right]_0^{\pi} = \frac{\pi^2}{3}, \text{ and} \\ a_n &= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx = \frac{2}{\pi} \left( \left[ \frac{1}{n} x^2 \sin nx \right]_0^{\pi} - \int_0^{\pi} \frac{2}{n} x \sin nx dx \right) \\ &= -\frac{2}{\pi} \left( - \left[ \frac{2}{n^2} x \cos nx \right]_0^{\pi} + \int_0^{\pi} \frac{2}{n^2} \cos nx dx \right) \\ &= \frac{4}{\pi n^2} \left[ x \cos nx \right]_0^{\pi} = \frac{4}{\pi n^2} \cdot \pi (-1)^n = \frac{4(-1)^n}{n^2}. \end{aligned}$$

Thus the solution is given by

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} a_n e^{-n^2 t} \cos nx = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} e^{-n^2 t} \cos nx \\ &= \frac{\pi^2}{3} - 4 \left( e^{-t} \cos x - \frac{1}{4} e^{-4t} \cos 2x + \frac{1}{9} e^{-9t} \cos 3x - \frac{1}{16} e^{-16t} \cos 4x + \dots \right). \end{aligned}$$

- (b) Give a fairly accurately sketch of the graphs of  $u = u(x, t)$  (in the  $xu$ -plane) for  $t = 0, \frac{1}{2}, 1, 10$ .

Solution: When  $t = 0$  we have  $u(x, 0) = x^2$ , when  $t = \frac{1}{2}$  we have  $u(x, \frac{1}{2}) \cong \frac{\pi^2}{3} - 4e^{-1/2} \cos x + \frac{4}{9}e^{-2} \cos 2x$ , when  $t = 1$  we have  $u(x, 1) \cong \frac{\pi^2}{3} - 4e^{-1} \cos x$ , and when  $t = 10$  we have  $u(x, 10) \cong \frac{\pi^2}{3}$ . These approximations are sketched below (in dark blue, lighter blue, cyan, and green):



4: (a) Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by  $f(z) = z^2$  and let  $v, w : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the real and imaginary parts of  $f$  so that  $f(x + iy) = v(x, y) + i w(x, y)$ . Show that  $v$  and  $w$  both satisfy Laplace's equation.

Solution: We have  $f(x + iy) = (x + iy)^2 = (x^2 - y^2) + i(2xy)$  and so  $v(x, y) = x^2 - y^2$  and  $w(x, y) = 2xy$ . We have  $\frac{\partial v}{\partial x} = 2x$  and  $\frac{\partial v}{\partial y} = -2y$  so that  $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 2 - 2 = 0$ , and we have  $\frac{\partial w}{\partial x} = y$  and  $\frac{\partial w}{\partial y} = x$  so that  $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 + 0 = 0$ .

(b) Solve Laplace's equation  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$  for  $u = u(x, y)$  on the square  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$  satisfying the boundary conditions  $u(x, 0) = x^2$ ,  $u(x, 1) = x - 1$ ,  $u(0, y) = -y^2$  and  $u(1, y) = 1 - y^2$ . Hint: use  $v(x, y)$  from Part (a) and notice that  $u(x, 1) \neq v(x, 1) = x^2 - 1$ .

Solution: Note that for the function  $v = v(x, y) = x^2 - y^2$  from Part (a) we have  $v(x, 0) = x^2$ ,  $v(x, 1) = x^2 - 1$ ,  $v(0, y) = -y^2$  and  $v(1, y) = 1 - y^2$ . These are similar to the desired boundary conditions for  $u = u(x, y)$ : indeed if  $u = u(x, y)$  satisfies the desired boundary conditions and we let  $w(x, y) = u(x, y) - v(x, y)$  then  $w$  will satisfy the boundary conditions  $w(x, 0) = 0$ ,  $w(x, 1) = x - x^2$ ,  $w(0, y) = 0$  and  $w(1, y) = 0$ .

Let us try to find a solution  $w = w(x, y)$  to Laplace's equation such that  $w(x, 0) = 0$ ,  $w(x, 1) = x - x^2$ ,  $w(0, y) = 0$  and  $w(1, y) = 0$ . We try  $w = XY$ . Laplace's equation becomes  $X''Y + XY'' = 0$ , that is  $\frac{X''}{X} = -\frac{Y''}{Y}$ , so that we must have  $\frac{X''}{X} = k = -\frac{Y''}{Y}$  for some constant  $k$ , and the initial conditions become  $X(x)Y(0) = 0$ ,  $X(x)Y(1) = x - x^2$ ,  $X(0)Y(y) = 0$  and  $X(1)Y(y) = 0$ , so that (for a non-zero solution) we must have  $X(0) = 0$ ,  $X(1) = 0$ ,  $Y(0) = 0$  and  $w(x, 1) = X(x)Y(1) = x - x^2$ . First we solve  $\frac{X''}{X} = k$ , that is  $X'' - kX = 0$ , with  $X(0) = 0$  and  $X(1) = 0$ . As with the wave equation, when  $k \geq 0$  there are no non-zero solutions, and when  $k = -\sigma^2$  with  $\sigma > 0$ , there are non-zero solutions only when  $k = -\sigma^2 = -(n\pi)^2$  and in this case the solutions are given by  $X_n = b_n \sin(n\pi x)$ . When  $k = -\sigma^2 = -(n\pi)^2$ , the second DE  $\frac{Y''}{Y} = -k$  becomes  $Y'' - (n\pi)^2 Y = 0$  which has solutions  $Y = Y_n = a_n e^{n\pi y} + b_n e^{-n\pi y}$  and the initial condition  $Y(0) = 0$  gives  $a_n + b_n = 0$  so that  $Y_n = a_n e^{n\pi y} - a_n e^{-n\pi y} = 2a_n \sinh(n\pi y)$ . Thus for each  $n \in \mathbb{Z}^+$  we have found a solution  $w = w_n(x, y) = X_n(x)Y_n(y) = c_n \sinh(n\pi y) \sin(n\pi x)$ , and this solution satisfies the 3 boundary conditions  $w_n(x, 0) = w_n(0, y) = w_n(1, y) = 0$ . We let

$$w = w(x, y) = \sum_{n=1}^{\infty} c_n \sinh(n\pi y) \sin(n\pi x).$$

To satisfy the last condition  $w(x, 1) = x - x^2$  we need  $\sum_{n=1}^{\infty} c_n \sinh(n\pi) \sin(n\pi x) = x - x^2$  for all  $0 \leq x \leq 1$ , so the numbers  $c_n \sinh(n\pi)$  must be equal to the Fourier coefficients of the odd periodic function of period 2 which is equal to  $f(x) = x - x^2$  for  $0 \leq x \leq 1$ . Thus we must have

$$\begin{aligned} c_n \sinh(n\pi) &= 2 \int_0^1 (x - x^2) \sin(n\pi x) dx = 2 \left( \left[ -\frac{1}{n\pi} (x - x^2) \sin(n\pi x) \right]_0^1 - \int_0^1 \frac{2}{n\pi} x \cos(n\pi x) dx \right) \\ &= -\frac{4}{n\pi} \int_0^1 x \cos(n\pi x) dx = -\frac{4}{n\pi} \left( \left[ \cos(n\pi x) \right]_0^1 - \int_0^1 \frac{1}{n\pi} \sin(n\pi x) dx \right) \\ &= \frac{4}{(n\pi)^2} \int_0^1 \sin(n\pi x) dx = -\frac{4}{(n\pi)^3} \left[ \frac{1}{n\pi} x \sin(n\pi x) \right]_0^1 = \frac{4}{(n\pi)^3} (1 - (-1)^n), \end{aligned}$$

so we have  $c_n \sinh(n\pi) = \frac{8}{(n\pi)^3}$  when  $n$  is odd and  $c_n = 0$  when  $n$  is even. Thus our solution  $w$  is given by

$$w(x, y) = \sum_{n \text{ odd}} \frac{8}{(n\pi)^3 \sinh(n\pi)} \sinh(n\pi y) \sin(n\pi x).$$

The solution  $u = u(x, y)$  to the original problem is given by  $u(x, y) = v(x, y) + w(x, y)$ , that is

$$u(x, y) = x^2 - y^2 + \sum_{n \text{ odd}} \frac{8}{(n\pi)^3 \sinh(n\pi)} \sinh(n\pi y) \sin(n\pi x).$$