9. Dimension

9.1 Definition: For an irreducible variety $X \subseteq \mathbf{R}^n$, we define the **dimension** of X to be

$$\dim(X) = \text{trans}_{\mathbf{F}} K(X).$$

For a reducible variety $X \subseteq \mathbf{F}^n$, we define the **dimension** of X to be the maximum of the dimensions of the irreducible components of X. We say that X has **pure dimension** when all of the irreducible components of X have the same dimension.

9.2 Example: When $a \in \mathbf{F}^n$ we have $\dim\{a\} = 0$ and, when $\mathbf{F}$ is infinite, $\dim(\mathbf{F}^n) = n$.

9.3 Theorem: *Let X and Y be irreducible affine varieties with $X \subseteq Y \subseteq \mathbf{F}^n$. Then $\dim(X) \leq \dim(Y)$ with $\dim(X) = \dim(Y) \iff X = Y$.*

Proof: Let $r = \dim(X) = \text{trans}_{\mathbf{F}} K(X)$. Reorder the variables x_k, if necessary, so that $\{x_1, \cdots, x_r\} \subseteq A(X) \subseteq K(X)$ is a transcendence basis for $K(X)$ over $\mathbf{F}$. Suppose, for a contradiction, that $\{x_1, \cdots, x_r\} \subseteq A(Y) \subseteq K(Y)$ is algebraically dependent over $\mathbf{F}$. Choose $0 \neq g \in \mathbf{F}[t_1, \cdots, t_r]$ such that $g(x_1, \cdots, x_r) = 0 \in A(Y) \subseteq K(Y)$. Then $g(x_1, \cdots, x_r)$ is equal to zero as a function from Y to $\mathbf{F}$. Since $X \subseteq Y$, it follows that $g(x_1, \cdots, x_r)$ is also equal to zero as a function from X to F, so we have $0 = g(x_1, \cdots, x_r) \in A(X) \subseteq K(X)$. But then $\{x_1, \cdots, x_r\} \subseteq K(X)$ is algebraically dependent, which contradicts the fact that $\{x_1, \cdots, x_r\}$ is a transcendence basis for $K(X)$ over $\mathbf{F}$. Thus $\{x_1, \cdots, x_r\} \subseteq K(Y)$ is algebraically independent, as claimed, and so $\dim(Y) = \text{trans}_{\mathbf{F}} K(Y) \geq r = \dim(X)$.

Suppose, for a contradiction, that $\dim(X) = \dim(Y) = r$ but that $X \subsetneqq Y$. Reorder the variables x_k, if necessary, so that $\{x_1, \cdots, x_r\} \subseteq A(X) \subseteq K(X)$ is a transcendence basis for $K(X)$ over $\mathbf{F}$. Then, as shown above, $\{x_1, \cdots, x_r\} \subseteq A(Y) \subseteq K(Y)$ is also a transcendence basis for $K(Y)$ over $\mathbf{F}$. Since $X \subsetneqq Y$ we have $I(Y) \subsetneqq I(X)$ so we can choose $u \in I(X)$ with $u \notin I(Y)$. Then we have $u = 0 \in A(X)$ and $u \neq 0 \in A(Y)$. Since $\{x_1, \cdots, x_r\}$ is a transcendence basis for $K(Y)$ over $\mathbf{F}$, it follows that u is algebraic over $\mathbf{F}(x_1, \cdots, x_r) \subseteq K(Y)$. Let p be the minimal polynomial for u over $\mathbf{F}(x_1, \cdots, x_r) \subseteq K(Y)$ multiplied by the least common denominator so that p is an irreducible polynomial in $\mathbf{F}[x_1, \cdots, x_r][t] \subseteq A(Y)[t]$. Write

$$p(t) = p_0(x_1, \cdots, x_r) + p_1(x_1, \cdots, x_r)t + \cdots p_\ell(x_1, \cdots, x_r)t^\ell.$$

Since $p(t)$ is irreducible in $A(Y)[t]$, we must have $0 \neq p_0(x_1, \cdots, x_r) \in A(Y)$ hence also $0 \neq p(x_1, \cdots, x_r) \in \mathbf{F}[x_1, \cdots, x_r]$ as a polynomial, because $\{x_1, \cdots, x_r\} \subseteq A(Y) \subseteq K(Y)$ is algebraically independent. Since $p(u) = 0 \in A(Y)$ and $X \subseteq Y$, we also have $p(u) = 0 \in A(X)$. Since $u = 0 \in A(X)$ we have $0 = p(u) = p_0(x_1, \cdots, x_r) \in A(X)$, hence also $0 = p(x_1, \cdots, x_r) \in \mathbf{F}[x_1, \cdots, x_r]$ as a polynomial, because $\{x_1, \cdots, x_r\} \subseteq A(X) \subseteq K(X)$ is algebraically independent. Thus we have obtained the desired contradiction.

9.4 Corollary: *Let $\mathbf{F}$ be an algebraically closed field and let $X \subseteq \mathbf{F}^n$ be an irreducible variety. The $\dim(X) \geq \ell$ where ℓ is the length of the longest chain of irreducible subvarieties $X_0 \subsetneqq X_1 \subsetneqq X_2 \subsetneqq \cdots \subsetneqq X_\ell = X$ or, equivalently, the length of the longest chain of prime ideals $0 = P_\ell \subsetneqq P_{\ell-1} \subsetneqq \cdots \subsetneqq P_1 \subsetneqq P_0 \subsetneqq A(X)$.*

9.5 Theorem: *(Hypersurfaces) Let* $\mathbf{F}$ *be an algebraically closed field, and let* $X \subseteq \mathbf{F}^n$ *be an irreducible variety. Then* $\dim(X) = n-1$ *if and only if* $X = V(f)$ *for some irreducible polynomial* $f \in \mathbf{F}[x_1, \cdots, x_n]$.

Proof: Suppose that $\dim(X) = n-1$. Since $X \subseteq \mathbf{F}^n$ and $\dim(X) \neq \dim(\mathbf{F}^n)$ we have $X \subsetneqq \mathbf{F}^n$ and so $\{0\} = I(\mathbf{F}^n) \subsetneqq I(X)$. Choose $0 \neq g \in I(X)$. Note that g is non-constant since $X \neq \emptyset$. Say $g = {p_1}^{k_1} \cdots {p_\ell}^{k_\ell}$ where the p_k are non-associate irreducible polynomials in $\mathbf{F}[x_1, \cdots, x_n]$. Since $f = {p_1}^{k_1} \cdots {p_k}^{k_\ell} \in I(X)$ and X is irreducible so that $I(X)$ is prime, it follows that $p_j \in I(X)$ for some index j, say $f = p_k \in I(X)$. Since $f \in I(X)$ we have $X \subseteq V(f)$. We claim that $X = V(f)$. Since f is irreducible and $\mathbf{F}$ is algebraically closed, it follows that $V(f)$ is irreducible. Since $X \subseteq V(f) \subsetneqq \mathbf{F}^n$ we have $n-1 = \dim(X) \leq \dim V(f) < \dim(\mathbf{F}^n) = n$ and so $\dim V(f) = n-1$. Since X and $V(f)$ are irreducible with $X \subseteq V(f)$ and $\dim(X) = \dim V(f)$, we have $X = V(f)$, as claimed.

Suppose, conversely, that $X = V(f)$ where $f \in \mathbf{F}[x_1, \cdots, x_n]$ is irreducible. Since $\mathbf{F}$ is algebraically closed, we know that $X = V(f)$ is irreducible and that $I(X) = \langle f \rangle$. One of the variables x_k must occur (in a term with a nonzero coefficient) in the polynomial f. Reorder the variables, if necessary, so that x_n occurs in f, say

$$f(x_1, \cdots, x_n) = f_0(x_1, \cdots, x_{n-1}) + f_1(x_1, \cdots, x_{n-1})x_n + \cdots + f_\ell(x_1, \cdots, x_{n-1}){x_n}^\ell.$$

Since $f \in I(X)$ so that $f = 0 \in A(X)$, we see that the element $x_n \in A(X)$ is a root of the polynomial $g(t) = f_0(x_1, \cdots, x_{n-1}) + \cdots + f_\ell(x_1, \cdots, x_{n-1})t^\ell$ and so x_n is algebraic over $\mathbf{F}(x_1, \cdots, x_{n-1}) \subseteq K(X)$, and hence we must have $\dim(X) \leq n-1$. Suppose, for a contradiction, that $\dim(X) < n-1$. Then $\{x_1, \cdots, x_{n-1}\} \subseteq A(X) \subseteq K(X)$ is algebraically dependent over $\mathbf{F}$ so we can choose a nonzero polynomial $0 \neq g \in \mathbf{F}[x_1, \cdots, x_{n-1}]$ such that $g(x_1, \cdots, x_{n-1}) = 0 \in A(X) \subseteq K(X)$ (so g is not zero as a polynomial, but g is zero as a function on X). Since $g = 0 \in A(X)$ we have $g \in I(X) = \langle f \rangle$ and so $f \big| g$ in the polynomial ring $\mathbf{F}[x_1, \cdots, x_n]$. This is not possible since x_n occurs in f but x_n does not occur in g. Thus we must have $\dim(X) = n-1$, as required.

9.6 Definition: A **hypersurface** in $\mathbf{F}^n$ is a variety $X \subseteq \mathbf{F}^n$ such that every irreducible component of X has dimension $n-1$. When $\mathbf{F}$ is algebraically closed, it follows from the above theorem that a hypersurface in $\mathbf{F}^n$ is any variety of the form $X = V(f) \subseteq \mathbf{F}^n$ for some non-constant polynomial $f \in \mathbf{F}[x_1, \cdots, x_n]$. In this case, when $f = {p_1}^{k_1}{p_2}^{k_2} \cdots {p_\ell}^{k_\ell}$ where the p_k are non-associate irreducible polynomials, the irreducible components of X are the varieties $X_k = V(p_k)$ and we have $I(X) = \sqrt{\langle f \rangle} = \langle p_1 p_2 \cdots p_\ell \rangle$.

9.7 Note: Recall (or verify) that, when R is an integral domain, two non-constant polynomials $f,g\in R[x]$ have a non-constant common factor when there exist non-zero polynomials $u,v\in R[x]$ with $\deg(v)<\deg(f)$ (or equivalently with $\deg(u)<\deg(g)$) such that $fu+gv=0\in R[x]$. Note that the equation $fu+gv=0$ can be written in matrix form as follows. If $f(x)=\sum\limits_{k=0}^{n}a_kx^k$ with $a_n\neq 0$ and $g(x)=\sum\limits_{k=0}^{m}b_kx^k$ with $b_m\neq 0$, and $u=\sum\limits_{k=0}^{m-1}u_kx^k$ and $v=\sum\limits_{k=0}^{n-1}$ then, considering u, v and $fu+gv$ as elements in R^m, R^n and R^{n+m}, the equation $fu+gv=0$ can be written as

$$\begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & & 0 & b_1 & b_0 & & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ a_n & & & a_0 & b_n & & & b_0 \\ 0 & a_n & & a_1 & 0 & b_n & & b_1 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_n & 0 & 0 & \cdots & b_n \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_m \\ v_0 \\ v_1 \\ \vdots \\ v_n \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The matrix on the left is denoted by $R_{n,m}(f,g)$. It is called the **resultant matrix** of f and g, and it has m columns involving the coefficients a_k of f and m columns involving the b_k of g. It follows from the above discussion that f and g have a non-constant common factor if and only if $\det R_{n,m}(f,g)=0$. We define the **resultant** of f and g to be

$$\operatorname{res}_{n,m}(f,g)=\det_{n,m}R(f,g).$$

Note that if $\deg(f)=n$ and $\deg(g)<m$ (or if $\deg(f)<n$ and $\deg(g)=m$) then it is still the case that f and g have a non-constant common factor if and only if $\operatorname{res}_{n,m}(f,g)=0$, but if $\deg(f)<n$ and $\deg(g)<m$ then $\operatorname{res}_{n,m}(f,g)=0$.

9.8 Note: Recall (or verify) that, when $\mathbf{F}$ is any field, a polynomial $f\in\mathbf{F}[x]$ has a repeated root (in its splitting field) if and only if f and its derivative f' have a non-constant common factor. From the above note, when $\deg(f)=\ell$ this occurs if and only if $\operatorname{res}_{\ell,\ell-1}(f,f')=0$. For $f\in\mathbf{F}[x]$ with $\deg(f)\leq\ell$ we define the degree ℓ **discriminant** of f to be

$$\operatorname{disc}_\ell(f)=\operatorname{res}_{\ell,\ell-1}(f,f').$$

Note that when $\deg(f)<\ell$ we have $\operatorname{disc}_\ell(f)=0$.

9.9 Definition: When X is an affine algebraic variety, we say that a property holds **generically** in X when the property holds at every point in some dense open set $U\subseteq X$. For example, when X and Y are affine varieties and $f:Y\to X$ is dominant polynomial or rational map, we say that f is generically $d:1$ when there is a dense open subset $U\subseteq X$ such that for every point $a\in U$ the fibre $f^{-1}(a)$ contains exactly d points.

9.10 Theorem: *Let* $\mathbf{F}$ *be an algebraically closed field, let* $X \subseteq \mathbf{F}^n$ *and* $Y \subseteq \mathbf{F}^m$ *be irreducible varieties, and let* $f : Y \to X$ *be a dominant polynomial map.*
(1) If $K(Y)$ *is transcendental over* $f^*\big(K(X)\big)$ *then* f *is generically* $\infty : 1$.
(2) If $K(Y)$ *is algebraic over* $f^*\big(K(X)\big)$ *then* f *is generically* $d : 1$ *where*

$$d = \big[K(Y) : f^*\big(K(X)\big)\big].$$

Proof: First let us consider a special case. Suppose that $X \subseteq \mathbf{F}^n$ and $Y \subseteq \mathbf{F}^{n+1}$ and that $f : Y \to X$ is the projection map given by $f(x, y) = x$ where $x = (x_1, x_2, \cdots, x_n)$. Let $u_k(x) = x_k \in A(X)$, let $v_k(x, y) = x_k \in A(Y)$, let $w(x, y) = y \in A(Y)$, and write $u = (u_1, \cdots, u_n)$ and $v = (v_1, \cdots, v_n)$. Then we have $A(X) = \mathbf{F}[u] = \mathbf{F}[u_1, \cdots, u_n]$ and $A(Y) = \mathbf{F}[v, w] = \mathbf{F}[v_1, \cdots, v_n, w]$. Note that the pullback $f^* : A(X) \to A(Y)$ is given by $f^*(g)(x, y) = g\big(f(x, y)\big) = g(x)$, so we have $f^*\big(A(X)\big) = \mathbf{F}[v]$ and the isomorphism $f^* : A(X) = \mathbf{F}[u] \to f^*\big(A(X)\big) = \mathbf{F}[v]$ is the natural map given by $f^*(u_k) = v_k$ for each index k.

Note that the element $w \in A(Y)$ can either be transcendental or algebraic over $f^*(K(X)) = \mathbf{F}(v)$. If w is transcendental over $\mathbf{F}(v)$ then $A(Y)$ is isomorphic to the polynomial ring $\mathbf{F}[v][y] \cong A(X)[y] \cong A(X \times \mathbf{F})$ and, in this case, we have $Y \cong X \times \mathbf{F}$ and, indeed we must have $Y = X \times \mathbf{F}$ and $f^{-1}(a) = \{a\} \times \mathbf{F}$ for all $a \in X$. Thus, in the case that w is transcendental over $f^*\big(K(X)\big) = \mathbf{F}(v)$, the map f is (globally) $\infty : 1$.

Suppose that w is algebraic over $\mathbf{F}(u)$. Let $p(v, y) = p(v)(y)$ be the minimal polynomial of $w \in A(Y)$ over $\mathbf{F}(v)$, multiplied by the least common denominator so that $p(v)(y) \in \mathbf{F}[v][y]$. The polynomial $p(v, y) \in \mathbf{F}[v][y]$ is given by (or extends to) a polynomial $p(x, y) \in \mathbf{F}[x, y]$. We claim that $Y = V(p) \cap (X \times \mathbf{F})$. Since $p(v, w) = 0 \in A(Y)$ it follows that $p(a, b) = 0$ for all $(a, b) \in Y$, and so we have $p \in I(Y)$ and hence $Y \subseteq V(p)$. Since the projection $f(x, y) = x$ maps X to Y, we also have $Y \subseteq X \times \mathbf{F}$, and so $Y \subseteq V(p) \cap (X \times \mathbf{F})$. Let $g = g(x, y) \in I(Y)$. Then, since $g = 0 \in A(Y)$ we have $g(v, w) = 0 \in A(Y)$, and so w is a root of $g(v)(y) \in \mathbf{F}[v][y]$. Since p is the minimal polynomial of w over $\mathbf{F}(v)$ it follows that $p(v)(y)\big|g(v)(y)$ in $\mathbf{F}[v][y]$. Using the isomorphism $f^* : \mathbf{F}[u] \to \mathbf{F}[v]$ we see that $p(u)(y)\big|g(u)(y)$ in $\mathbf{F}[u][y] = A(X)[y]$, say $g(u)(y) = p(u)(y)k(u)(y) \in A(X)[y] \cong A(X \times \mathbf{F})$. Represent $k(u)(y)$ by a polynomial $k(x, y)$ and note that $g(x, y) - p(x, y)k(x, y) \in I(X \times F)$. Thus we have $g \in \langle p \rangle + I(X \times \mathbf{F})$. This shows that $I(Y) \subseteq \langle p \rangle + I(X \times \mathbf{F})$. It follows that $V(p) \cap (X \times \mathbf{F}) = V\big(\langle p \rangle + I(X \times \mathbf{F})\big) \subseteq Y$. Thus $Y = V(p) \cap (X \times \mathbf{F})$, as claimed.

Write $p(x, t) = p_0(x) + p_1(x) + \cdots + p_\ell(x)t^\ell$ with each $p_k(x) \in \mathbf{F}[x]$ and $p_\ell(x) \notin I(X)$. Since $Y = V(p)$, for each point $a \in X = \mathbf{F}^n$, the fiber $f^{-1}(a)$ is equal to the set of all pairs $(a, b) \in \mathbf{F}^{n+1}$ for which b is a root of the polynomial $p(a)(t) \in \mathbf{F}[t]$. When $\mathbf{F}$ is algebraically closed, the fiber $f^{-1}(a)$ contains exactly ℓ points when the polynomial $p(a)(t)$ has exactly ℓ distinct roots, that is when $\operatorname{disc}_\ell\big(p(a)\big) \neq 0$. Let $U = \big\{x \in X \big| \operatorname{res}_\ell\big(p(x)\big) \neq 0\big\}$. Note that $\operatorname{disc}_\ell\big(p(x)\big)$ is a polynomial in $x = (x_1, \cdots, x_n)$ so U is open in X. Also note that $U \neq \emptyset$ because if we had $\operatorname{res}_\ell\big(p(x)\big) = 0$ for all $x \in \mathbf{F}^n$ then we would have $\operatorname{res}_\ell\big(p(u)\big) = 0 \in A(X) = \mathbf{F}[u]$ hence $\operatorname{res}_\ell\big(p(v)\big) = 0 \in \mathbf{F}[v]$, but then $p(v)(y)$ and $p(v)'(y)$ would have a non-constant common factor in $\mathbf{F}[v][t]$, and hence $p(v)(y)$ would be reducible. Thus, in the case that w is algebraic over $f^*\big(K(X)\big) = \mathbf{F}(v)$, the map f is generically $\ell : 1$ where

$$\ell = \deg p(v)(y) = \big[\mathbf{F}(v)[y] : \mathbf{F}(v)\big] = \big[K(Y) : f^*\big(K(X)\big)\big].$$

This completes the proof of the theorem in the special case that $X \subseteq \mathbf{F}^n$ and $Y \subseteq \mathbf{F}^{n+1}$ and $f : Y \to X$ is the projection map $f(x, y) = x$.

Now consider the general case. Suppose $X \subseteq \mathbf{F}^n$, $Y \subseteq \mathbf{F}^m$ and $f : Y \to X$ is any dominant polynomial map. Let $u_k(x) = x_k \in A(X)$ so that $A(X) = \mathbf{F}[u] = \mathbf{F}[u_1, \cdots, u_n]$. Let $w_k(y) = y_k \in A(Y)$ so that $A(Y) = \mathbf{F}[w_1, \cdots, w_m]$. Say $f(y) = \big(f_1(y), \cdots, f_n(y)\big)$ and let $v_k(y) = f_k(y) \in A(Y)$. Note that $f^*(u_k)(y) = u_k\big(f(y)\big) = f_k(y) = v_k(y)$ so we have $f^*\big(A(X)\big) = f^*\big(\mathbf{F}[u]\big) = \mathbf{F}[v]$ and the map $f^* : \mathbf{F}[u] \to \mathbf{F}[v]$ is given by $f^*(u_k) = v_k$. We have $A(Y) = \mathbf{F}[w_1, \cdots, w_m] = f^*\big(A(X)\big)[w_1, \cdots, w_m] = \mathbf{F}[v_1, \cdots, v_n][w_1, \cdots, w_m]$.

Suppose that $A(Y)$ is algebraic over $f^*\big(A(X)\big)$. We have a tower of integral domains

$$AX \cong f^*\big(A(X)\big) = \mathbf{F}[v] \subseteq \mathbf{F}[v][w_1] \subseteq \mathbf{F}[v][w_1, w_2] \subseteq \cdots \subseteq \mathbf{F}[v][w_1, \cdots, w_m] = A(Y).$$

At each stage, w_k is algebraic over the previous quotient field $\mathbf{F}(v)[w_1, \cdots, w_{k-1}]$. We obtain, correspondingly, a chain of irreducible varieties $X_k \subseteq \mathbf{F}^{n+k}$ and projection maps $f_k : X_k \to X_{k-1}$, as in the special case studied above,

$$X = X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_{m-1} \longleftarrow X_m \cong Y.$$

At each stage, we have $X_k = V(p_k) \cap (X_{k-1} \times \mathbf{F})$ where $p_k \in \mathbf{F}[x_1, \cdots, x_n, y_1, \cdots, y_k]$ is a polynomial such that $p_k(v, w_1, \cdots, w_{k-1}, y)$ is the minimal polynomial of w_k over $\mathbf{F}(v)[w_1, \cdots, w_{k-1}]$, and we have a natural isomorphism $\phi_k : A(X_k) \to \mathbf{F}[v][w_1, \cdots, w_k]$ of $\mathbf{F}$-algebras. The given dominant polynomial map $f : Y \to X$ is equal to the composite $f = f_1 \circ f_2 \circ \cdots \circ f_m \circ g$ where $g : Y \to X_m$ is the polynomial isomorphism for which $g^* = \phi_m : A(X_m) \to A(Y)$. Each projection map f_k is generically $\ell_k : 1$ where

$$\ell_k = \deg(p_k) = \big[\mathbf{F}(v)[w_1, \cdots, w_k] : \mathbf{F}(v)[w_1, \cdots, w_{k-1}]\big],$$

and we have $d = \big[K(Y) : f^*\big(K(X)\big)\big] = \ell_1 \ell_2 \cdots \ell_m$.

We need to show that the composite $f_1 \circ f_2 \circ \cdots \circ f_m$ is generically $d : 1$. By induction, it suffices to show that $f_k \circ f_{k+1}$ is generically $\ell_k \ell_{k+1} : 1$. Let $U_{k-1} \subseteq X_{k-1}$ and $U_k \subseteq X_k$ be dense open subsets such that ${f_k}^{-1}(a)$ contains exactly ℓ_k points for every $a \in U_{k-1}$ and ${f_{k+1}}^{-1}(b)$ contains exactly ℓ_{k+1} points for every $b \in U_k$. Let ${U_k}^c = X_k \setminus U_k$ and note that ${U_k}^c$ is a closed subvariety of X_k. For $a \in U_{k-1} \setminus f({U_k}^c)$, the fibre ${f_k}^{-1}(a)$ contains exactly ℓ_k points and we have ${f_k}^{-1}(a) \subseteq U_k$ and so the set ${f_{k+1}}^{-1}\big({f_k}^{-1}(a)\big)$ contains exactly $\ell_k \ell_{k+1}$ points. Thus it suffices to show that $X_{k-1} \setminus f({U_k}^c)$ contains a dense open set in X_{k-1} or, equivalently, to show that $\overline{f({U_k}^c)} \subsetneqq X_{k-1}$. We do this by comparing dimensions. First note that since w_k is algebraic over $\mathbf{F}(v)[w_1, \cdots, w_{k-1}]$ we have

$$\dim(X_k) = \operatorname{trans}_{\mathbf{F}} \mathbf{F}(v)[w_1, \cdots, w_k] = \operatorname{trans}_{\mathbf{F}} \mathbf{F}(v)[w_1, \cdots, w_{k-1}] = \dim(X_{k-1}).$$

Since $U_k \neq \emptyset$ we have ${U_k}^c \subsetneqq X_k$ so $\dim({U_k}^c) < \dim(X_k) = \dim(X_{k-1})$. Since the map $f_k : {U_k}^c \to f_k({U_k}^c)$ is surjective, it follows that the map $f_k : {U_k}^c \to \overline{f_k({U_k}^c)}$ is dominant, so we have $\dim\big(\overline{f_k({U_k}^c)}\big) \leq \dim({U_k}^c) < \dim(X_{k-1})$, and hence $\overline{f_k({U_k}^c)} \subsetneqq X_{k-1}$, as required.

Suppose, finally, that $K(Y)$ is transcendental over $f^*\big(K(X)\big)$. Reorder the variables, if necessary, so that $\{w_1, \cdots, w_r\}$ is a transcendence basis for $K(Y) = \mathbf{F}(v)(w_1, \cdots, w_m)$ over $f^*\big(K(X)\big) = \mathbf{F}(v)$. Form the corresponding chain of varieties X_k and projection maps $f_k : X_k \to X_{k-1}$, as above. For $1 \leq k \leq r$ we have $X_k = X_{k-1} \times \mathbf{F}$ and f_k is (globally) $\infty : 1$, and for $r < k \leq m$ the map f_k is generically $\ell_k : 1$. Thus the composite $f_1 \circ \cdots \circ f_r$ is (globally) $\infty : 1$ and the composite $f_{r+1} \circ \cdots \circ f_m$ is generically $d : 1$ where $d = \ell_{r+1} \ell_{r+2} \cdots \ell_m$, and so $f_1 \cdots \circ f_m$ (hence also the original dominant map f) is generically $\infty : 1$.

9.11 Corollary: *Let* $\mathbf{F}$ *be algebraically closed, let* $X \subseteq \mathbf{F}^n$ *and* $Y \subseteq \mathbf{F}^m$ *be irreducible varieties, and let* $f : Y \to X$ *be a dominant polynomial map. If* $A(X)$ *is integral over* $f^*\big(A(Y)\big)$ *then* f *is surjective and finite*$:1$.

Proof: This result can be extracted from the proof of the above theorem. Using the notation of the proof, in the case that $X \subseteq \mathbf{F}^n$, $Y \in \mathbf{F}^{n+1}$, $A(X) = \mathbf{F}[u] = \mathbf{F}[u_1, \cdots, u_n]$, and $A(Y) = \mathbf{F}[v] = \mathbf{F}[v_1, \cdots, v_n, w]$, recall that $Y = V(p) \cap (X \times \mathbf{F}) \subseteq \mathbf{F}^{n+1}$ where $p(x, y) \in \mathbf{F}[x, y]$ and $p(v)(y)$ is the minimal polynomial of w over $f^*\big(A(X)\big) = \mathbf{F}(v)$. For $p(x, t) = p_0(x) + p_1(x)t + \cdots + p_\ell(x)t^\ell$ with $p_\ell(v) \neq 0 \in \mathbf{F}[v]$, if $A(Y)$ is integral over $f^*\big(A(X)\big) = \mathbf{F}[v]$, then we have $p_\ell(v) = 1 \in \mathbf{F}[v]$ and hence $p_\ell(u) = 1 \in \mathbf{F}[u] = A(X)$. Then for all $a \in X$ we have $p_\ell(a) = 1$, so the polynomial $p(a) \in \mathbf{F}[y]$ is of degree ℓ, for every $a \in X$. It follows that the fibre $f^{-1}(a) = \big\{(a, y)\big|p(a)(y) = 0\big\}$ always has at least 1 and at most ℓ elements, for every $a \in X$.

9.12 Corollary: *Let* $\mathbf{F}$ *be an algebraically closed and let* X *be an irreducible affine variety. Then* $\dim(X) = d$ *if and only if there exists a surjective finite*$:1$ *polynomial map* $f : X \to \mathbf{F}^d$ *if and only if there exists a dominant generically finite*$:1$ *polynomial map* $f : X \to \mathbf{F}^d$.

Proof: Suppose that $\dim(X) = d$. By Noether's Normalization Lemma, we can choose $u_1, \cdots, u_d \in A(X)$ such that $\{u_1, \cdots, u_d\}$ is algebraically independent (so it is a transcendence basis for $K(X)$ over $\mathbf{F}$) and $A(X)$ is integral over $\mathbf{F}[u_1, \cdots, u_d]$. Since $\{u_1, \cdots, u_d\}$ is algebraically independent, the $\mathbf{F}$-algebra homomorphism $\phi : \mathbf{F}[t_1, \cdots, t_d] \to AX$ given by $\phi(t_k) = u_k$ is injective. Let $f : X \to \mathbf{F}^d$ be the dominant polynomial map with $f^* = \phi$. Since AX is integral over $f^*\big(\mathbf{F}[t_1, \cdots, t_d]\big) = \mathbf{F}[u_1, \cdots, u_d]$ it follows from the above theorem that f is surjective and finite$:1$.

Suppose, on the other hand, that $f : X \to \mathbf{F}^d$ is a dominant and generically finite$:1$ polynomial map. Then, by the above theorem, $K(\mathbf{F}^d)$ is algebraic over $f^*\big(K(X)\big)$ and so

$$\dim(X) = \text{trans}_{\mathbf{F}} K(X) = \text{trans}_{\mathbf{F}} f^*\big(K(X)\big) = \text{trans}_{\mathbf{F}} K(\mathbf{F}^d) = \dim(\mathbf{F}^d) = d.$$

9.13 Corollary: *Let* $\mathbf{F}$ *be an algebraically closed field, and let* X *and* Y *be affine varieties with* Y *irreducible and with* $X \subsetneqq Y$*. Then there exists an irreducible variety* Z *with* $X \subseteq Z \subsetneqq Y$ *such that* $\dim(Z) = \dim(Y) - 1$.

Proof: Let $d = \dim Y$*. Since every irreducible component of* X *is a proper subvariety of* Y*, we have* $\dim X < \dim Y = d$*. Let* $f : Y \to \mathbf{F}^d$ *be a surjective and finite*$:1$ *polynomial map. Then* $f : X \to f(X) \subseteq \mathbf{F}^d$ *is surjective and finite*$:1$*, and so* $f : X \to \overline{f(X)}$ *is dominant and generically finite*$:1$*. It follows that* $\dim \overline{f(X)} = \dim(X) < d$ *and so* $\overline{f(X)} \subsetneqq \mathbf{F}^d$*. Since* $\overline{f(X)} \subsetneqq \mathbf{F}^d$ *we have* $\{0\} = I(\mathbf{F}^d) \subsetneqq I\big(\overline{f(X)}\big)$ *so we can choose* $0 \neq g \in I\big(\overline{f(X)}\big)$*. Then we have* $\overline{f(X)} \subseteq V(g) \subsetneqq \mathbf{F}^d$*. Let* $Z = f^{-1}\big(V(g)\big) \subseteq Y$*. Since* $f : Z \to f(Z) = V(g)$ *is surjective and finite*$:1$*, we have* $\dim Z = \dim V(g) = d - 1$*. Since* $f(X) \subseteq \overline{f(X)} \subseteq V(g)$ *we have* $X = f^{-1}\big(f(X)\big) \subseteq f^{-1}\big(V(g)\big) = Z$.

9.14 Corollary: *Let* $\mathbf{F}$ *be an algebraically closed field, and let* X *be an irreducible affine variety. Then* $\dim(X)$ *is equal to the length* ℓ *of the longest chain of irreducible varieties* $\emptyset \neq X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_\ell = X$ *or, equivalently, the length* ℓ *of the longest chain of prime ideals* $\{0\} = P_\ell \subsetneqq P_{\ell-1} \subsetneqq \cdots \subsetneqq P_1 \subsetneqq P_0 \subsetneqq A(X)$.

Proof: This follows from the above corollary together with Corollary 9.4.