6. Rational Maps and Birational Equivalence

6.1 Definition: Let $X \subseteq \mathbf{F}^n$ be an irreducible variety. Let $p, q \in \mathbf{F}[x_1, \cdots, x_n]$ with $q \notin I(X)$. Alternatively let $p, q \in A(X)$ with $q \neq 0$. Then p/q defines a map $p/q : U_q \to \mathbf{F}$, where $U_q = \{x \in X \big| q(x) \neq 0\}$. We consider two such maps $p/q : U_q \to \mathbf{F}$ and $r/s : U_s \to \mathbf{F}$ to be equivalent when $\dfrac{p(x)}{q(x)} = \dfrac{r(x)}{s(x)}$ for all $x \in U_q \cap U_s$. Notice that since X is irreducible, we have $\overline{U_q \cap U_s} = X$ so p/q is equivalent to $r/s \iff p(x)s(x) = q(x)r(x)$ for all $x \in U_q \cap U_s \iff ps - qr \in I(U_q \cap U_s) = I(X) \iff ps - qr = 0 \in A(X)$. An equivalence class f of such maps determines a map $f : U \subseteq X \to \mathbf{F}$, where $U = \bigcup\limits_{p/q \in f} U_q$. Such a map is called a **rational map** on X, and the set U is called the **domain** of f. We use the notation $f : X \to \mathbf{F}$ for a rational map on X (even when the domain of f is not all of X), and we write p/q for the rational map determined by the equivalence class of p/q. The set of rational maps, denoted by $K(X)$, is a field, called the **field of rational maps** on X (or the **function field**) of X, which we can identify with the quotient field of the integral domain $A(X)$.

$$\begin{aligned} K(X) &= \left\{ \frac{p}{q} \middle| p, q \in \mathbf{F}[x_1, \cdots, x_n], q \notin I(X) \right\} \text{ with } \frac{p}{q} = \frac{r}{s} \in K(X) \iff ps - qr \in I(X) \\ &= \left\{ \frac{p}{q} \middle| p, q \in A(X), q \neq 0 \right\} \text{ with } \frac{p}{q} = \frac{r}{s} \in K(X) \iff ps - qr = 0 \in A(X) \end{aligned}.$$

For $a \in X$ we say that f is **defined** at a (or that f is **regular** at a) when a is in the domain $U = \bigcup\limits_{p/q = f} U_q$, that is when f can be written in the form $f = p/q$ for some p, q with $q(a) \neq 0$. Otherwise we say that f has a **pole** at a. Note that the pole set of f is the closed set $X \setminus U = \bigcap\limits_{p/q = f} V(q) \cap X$. We say that f has a **zero** at a when f is defined at a and can be written in the form $f = p/q$ for some p, q with $q(a) \neq 0$ and $p(a) = 0$. The zero set of f is not always a subvariety of X, but it is a closed subset of the domain U. On the other hand, note that if f can be written as $f = p/q$ with $p(a) \neq 0$ and $q(a) = 0$, then f must have a pole at a since if not, then we could write $f = r/s$ where $s(a) \neq 0$, and then since $p/q = r/s \in K(X)$ we would have $p(a)s(a) = r(a)q(a)$, but $p(a)s(a) \neq 0$ while $r(a)q(a) = 0$.

6.2 Example: If $\mathbf{F}$ is an infinite field, we have seen that $A(\mathbf{F}^n) = \mathbf{F}[x_1, \cdots, x_n]$, so $K(\mathbf{F}^n) = \mathbf{F}(x_1, \cdots, x_n)$. At the other extreme, if $a \in \mathbf{F}^n$ then $A(\{a\}) = K(\{a\}) = \mathbf{F}$.

6.3 Example: Let $\mathbf{F}$ be an infinite field. For each of the varieties $V(y - x^2)$, $V(y^2 - x^3)$, $V(y^2 - x^3 - x^2)$ and $V(x^2 + y^2 - y)$, show that the variety is irreducible in $\mathbf{F}^2$, then find the pole set of the rational map $g(x, y) = y/x$.

6.4 Example: Let X be the parabola $X = V(y - x^2) \subseteq \mathbf{F}^2$. Then X is irreducible since $y - x^2$ is irreducible in $\mathbf{F}[x, y]$ and since X is infinite. The map $g = \frac{y}{x}$ is defined at all points $(x, y) \in X$ with $x \neq 0$, that is everywhere except perhaps at $(0, 0)$. But since $y = x^2 \in K(X)$, we have $g = \frac{y}{x} = \frac{yx}{x^2} = \frac{yx}{y} = x \in K(X)$, so g is also defined at $(0, 0)$. Indeed this shows that g is actually a polynomial map on X.

Now let X be the circle $X = V(x^2 + y^2 - y)$. Note that for every $t \in \mathbf{F}$ we have $\left(\frac{t}{1+t^2}, \frac{t^2}{1+t^2}\right) \in X$ and so X is infinite. Also $x^2 + y^2 - y$ is irreducible in $\mathbf{F}[x, y]$, and so

X is irreducible. The map $g = \frac{y}{x}$ is defined at all points $(x,y) \in X$ with $x \neq 0$, that is everywhere except possibly at $(0,0)$ and at $(0,1)$. At the point $(0,1)$, we have $y \neq 0$ and $x = 0$ so g must have a pole at $(0,1)$. On the other hand, since $x^2 = y - y^2 \in K(Y)$, we have $g = \frac{y}{x} = \frac{yx}{x^2} = \frac{yx}{y-y^2} = \frac{x}{1-y} \in K(Y)$. Thus g is defined at $(0,0)$, indeed $f(0,0) = 0$.

Next, let X be the cusp curve $X = V(y^2 - x^3)$. Note that for every $t \in \mathbf{F}$, we have $(t^2, t^3) \in X$ so X is infinite. Also, $y^2 - x^3$ is irreducible in $\mathbf{F}[x,y]$, and so X is irreducible, and $I(X) = \langle y^2 - x^3 \rangle$. The map $g = y/x$ is defined everywhere except perhaps when $x = 0$, that is at the point $(0,0)$. We claim that g is not regular at $(0,0)$. Suppose, for a contradiction, that $g = p/q$ with $q(0,0) \neq 0$. Then we would have $\frac{y}{x} = \frac{p}{q} \in K(X)$ so $yq - xp \in I(X) = \langle y^2 - x^3 \rangle$, say $y\,q(x,y) - x\,p(x,y) = (y^2 - x^3)\,k(x,y)$, where $k \in \mathbf{F}[x,y]$. But notice that since $q(0,0) \neq 0$, the coefficient of y in $yq - xp$ is non-zero, but the coefficient of y in $(y^2 - x^3)\,k$ is zero.

Finally, let X be the alpha curve $X = V(y^2 - x^3 - x^2)$. For each $t \in \mathbf{F}$, we have $\big(t^2 - 1, t(t^2 - 1)\big) \in X$, so X is infinite. Also, $y^2 - x^3 - x^2$ is irreducible in $\mathbf{F}[x,y]$, so X is irreducible and $I(X) = \langle y^2 - x^3 - x^2 \rangle$. The map $g = y/x$ is defined everywhere except perhaps at $(0,0)$. We claim that g is not regular at $(0,0)$. We could prove this as we did for the cusp curve, but it is amusing to give a more geometric argument, even though the argument only works in the case that $\mathrm{char}\,(\mathbf{F}) \neq 2$. Geometric intuition tells us that $g = y/x$ cannot be regular at $(0,0)$ because y/x is the slope of the vector (x,y) but the alpha curve has two slopes at $(0,0)$. To express this idea precisely, we define $f : \mathbf{F} \to X$ by $f(t) = \big(t^2 - 1, t(t^2 - 1)\big)$. Note that $f(-1) = f(1) = (0,0)$ and that $g(f(t)) = \frac{t(t^2-1)}{t^2-1} = t$. If g were defined at $(0,0)$, then we would have $-1 = g(f(-1)) = g(0,0) = g(f(1)) = 1$.

6.5 Definition: Let $X \subseteq \mathbf{F}^n$ and $Y \subseteq \mathbf{F}^m$ be irreducible varieties. Let $f = (f_1, \cdots, f_m) \in K(X)^m$. Then f defines a map $f : U \to \mathbf{F}^m$ whose domain U is the intersection of the domains of the rational maps f_i. If $f(U) \subseteq Y$, then we say that f is a **rational map from** X **to** Y, and we write $f : X \to Y$. Note that we can write each f_i as $f_i = p_i/q$ (using a common denominator q) and so we can write $f = p/q$ where $p \in A(X)^m$ and $0 \neq q \in A(X)$. For $a \in X$ we say that f is **defined** at a (or that f is **regular** at a) when f can be written in the form $f = p/q$ for some p, q with $q(a) \neq 0$. Otherwise we say that f has a **pole** at a. The **domain** of f, that is the above set U, is the set of all points in X at which f is defined; U is an open dense subset of X. The **pole set** of f, which is the set of points in X at which f has a pole, is the proper subvariety $X \setminus U$. The set $f(U)$ is called the **range** (or **image**) of f.

6.6 Note: Let $f : X \to Y$ be a rational map of irreducible affine varieties with domain $U \subseteq X$. Then for any subvariety $Z \subseteq Y$, the inverse image $f^{-1}(Z) = \{x \in U | f(x) \in Z\}$ is closed in U. In other words, f is continuous.

Proof: Let Z be a subvariety of $Y \subseteq \mathbf{F}^m$. Say $Z = V(S)$ where $S \subseteq \mathbf{F}[y_1, \cdots, y_m]$. Then for $x \in U$ we have

$$\begin{aligned} x \in f^{-1}(Z) &\iff p(x)/q(x) \in Z \text{ for all } p,q \text{ with } f = p/q \text{ and } q(x) \neq 0 \\ &\iff g\big(\tfrac{p(x)}{q(x)}\big) = 0 \text{ for all } g \in S \text{ and all } p,q \text{ with } f = p/q,\ q(x) \neq 0 \\ &\iff g\big(\tfrac{p(x)}{q(x)}\big) q(x)^{\deg g} \text{ for all } g \in S \text{ and all } p,q \text{ with } f = p/q,\ q(x) \neq 0 \end{aligned} .$$

Write $h_{g,p,q}(x) = g\big(\frac{p(x)}{q(x)}\big) q(x)^{\deg g} \in \mathbf{F}[x_1, \cdots, x_n]$. Notice that if $f = p/q = r/s \in K(X)$ where $q(a) \neq 0$ (and perhaps $s(a) = 0$) then $h_{g,p,q}(x) = h_{g,r,s}(x)$ for all $x \in U_q \cap U_s$ and hence for all $x \in X$. So $f^{-1}(Z) = U \cap V(T)$ where $T = \big\{h_{g,p,q} \big| g \in S \text{ and } f = p/q\big\}$.

6.7 Definition: Given rational maps $f : X \to Y$ and $g : Y \to Z$ of irreducible varieties with domains U and V respectively, it is not always possible to compose them, because the range $f(U)$ may be disjoint from the domain V of g. If $f(U)$ is not disjoint from V, then we define the **composite** $g \circ f : X \to Z$ as follows. Choose any point $a \in f^{-1}(V)$, and write $f = p/q$ with $q(a) \neq 0$ and write $g = r/s$ with $s(a) \neq 0$. Then for all $x \in U_q \cap f^{-1}(V_s)$, which is open and dense in X (it is non-empty since it contains a and it is open by the above note, so it is dense since X is irreducible), we have

$$g(f(x)) = \frac{r\left(\frac{p(x)}{q(x)}\right)}{s\left(\frac{p(x)}{q(x)}\right)} = \frac{r\left(\frac{p(x)}{q(x)}\right)q(x)^d}{s\left(\frac{p(x)}{q(x)}\right)q(x)^d} = \frac{u(x)}{v(x)}$$

where $d = \max\{\deg r, \deg s\}$, $u(x) = r\left(\frac{p(x)}{q(x)}\right)q(x)^d$ and $v(x) = s\left(\frac{p(x)}{q(x)}\right)q(x)^d$. We define $g \circ f$ to be the rational map u/v.

6.8 Definition: Let X and Y be irreducible affine varieties. A **rational isomorphism** (or a **birational map**) from X to Y is a rational map $f : X \to Y$ which has a rational inverse, that is a rational map $g : Y \to X$ such that $g \circ f : X \to X$ is the identity on X and $f \circ g : Y \to Y$ is the identity on Y. We say that X and Y are **birationally isomorphic** (or simply **birational**), and we write $X \sim Y$, if there exists a rational isomorphism from X to Y.

6.9 Definition: A rational map $f : X \to Y$ of irreducible affine varieties is called **dominant** if the range of f is dense in Y.

6.10 Note: Let $f : X \to Y$ be a rational map of irreducible affine varieties.
(1) If f is dominant then f can be composed with any rational map $g : Y \to Z$.
(2) If $f : X \to Y$ is a rational isomorphism, then it must be dominant.

Proof: We leave the proof of (1) as an exercise and prove part (2). Suppose that $f : X \to Y$ is a rational isomorphism with rational inverse $g : Y \to X$. Let U be the domain of f and let V be the domain of g. Since g is continuous and U is open, $g^{-1}(U) = \{y \in V | g(y) \in U\}$ is open in V hence also in Y, and since the composite $f \circ g$ exists, $g^{-1}(U)$ is nonempty. So, since Y is irreducible, $g^{-1}(U)$ is dense in Y. For $y \in g^{-1}(U)$ we have $g(y) \in U$ so f is defined at $g(y)$ and $y = f(g(y))$ is in the range of f. Thus the range of f contains $g^{-1}(U)$, which is dense in Y.

6.11 Note: Let $f : X \to Y$ be a rational map of irreducible varieties with domain U. Then $\overline{f(U)}$ is irreducible.

Proof: Suppose, for a contradiction, that $\overline{f(U)}$ is reducible, say $\overline{f(U)} = Z \cup W$ with $Z, W \subsetneqq \overline{f(U)}$ both closed. Note that $f(U) \subseteq Z \cup W$, so $U = f^{-1}(Z) \cup f^{-1}(W)$ and hence $X = \overline{U} = \overline{f^{-1}(Z)} \cup \overline{f^{-1}(W)}$. Also, we have $f(U) \not\subseteq Z$ (otherwise we would have $\overline{f(U)} \subseteq \overline{Z} = Z$), so $f^{-1}(Z) \neq U$. Since $f^{-1}(Z)$ is closed in U, we have $\overline{f^{-1}(Z)} \cap U = f^{-1}(Z) \cap U = f^{-1}(Z) \neq U$, and so $\overline{f^{-1}(Z)} \neq X$. Similarly $\overline{f^{-1}(W)} \neq X$, so X is reducible.

6.12 Definition: Many useful rational maps can be obtained using the **projection from a point** in $\mathbf{F}^n$ to a hyperplane in $\mathbf{F}^n$ (that is, to an affine space of dimension $n-1$). The projection in $\mathbf{F}^2$ from the origin to the line $x=1$ is given by $g(x,y)=(1,y/x)$. More generally, the projection from $0\in\mathbf{F}^n$ to the hyperplane $x_k=1$ is given by

$$g(x_1,\cdots,x_n)=\left(\frac{x_1}{x_k},\cdots,\frac{x_k}{x_k},\cdots,\frac{x_n}{x_k}\right).$$

6.13 Example: Let $\mathbf{F}$ be an infinite field. When X is any one of the varieties $V(y-x^2)$, $V(y^2-x^3)$, $V(y^2-x^3-x^2)$ or $V(y^2+x^2-x)$, the projection in $\mathbf{F}^2$ from the origin to the line $x=1$, given by $g(x,y)=\frac{y}{x}$, gives a birational map $g:X\to\mathbf{F}$. In each case, the inverse $f:\mathbf{F}\to X$ gives a familiar parametric equation for the curve. Verify that when $X=V(y-x^2)$ we have $f(t)=(t,t^2)$, when $X=V(y^2-x^3)$ we have $f(t)=(t^2,t^3)$, when $X=V(y^2-x^3-x^2)$ we have $f(t)=\big(t^2-1,t(t^2-1)\big)$, and when $X=V(y^2+x^2-x)$ we have $f(t)=\left(\frac{1}{t^2+1},\frac{t}{t^2+1}\right)$.

6.14 Example: Let $\mathbf{S}^2$ be the unit sphere $\mathbf{S}^2=V(x^2+y^2+z^2-1)\subseteq\mathbf{R}^2$. Use the **stereographic projection** from the north pole, that is the projection from the point $(0,0,1)$ to the plane $z=0$, to show that $\mathbf{S}^2$ is irreducible and that $\mathbf{S}^2\sim\mathbf{R}^2$.

Solution: Given $(x,y,z)\in\mathbf{R}^3$ with $z\neq 1$, the line from $(0,0,1)$ to (x,y,z) is given by $\alpha(t)=(0,0,1)+(x,y,z-1)\,t$. The point of intersection with the plane $z=0$ is given by $1+(z-1)t=0$, that is $t=\frac{1}{1-z}$. Thus the projection g from $(0,0,1)$ to the plane $z=0$ is given by $g(x,y,z)=\left(\frac{x}{1-z},\frac{y}{1-z}\right)$.

On the other hand, given a point $(u,v,0)$ in the plane $z=0$, the line from $(0,0,1)$ to $(u,v,0)$ is given by $\beta(t)=(0,0,1)+(u,v,-1)\,t=(ut,vt,1-t)$, and we have $\beta(t)\in\mathbf{S}^2\iff(ut)^2+(vt)^2+(1-t)^2=1\iff u^2t^2+v^2t^2+1-2t+t^2=1\iff t\,(u^2t+v^2t-2+t)=0\iff t=0$ or $t=\frac{2}{u^2+v^2+1}$. We have $\beta(0)=(0,0,1)$, so we define a rational map $f:\mathbf{R}^2\to\mathbf{R}^3$ by $f(u,v)=\beta\left(\frac{2}{u^2+v^2+1}\right)=\left(\frac{2u}{u^2+v^2+1},\frac{2v}{u^2+v^2+1},\frac{u^2+v^2-1}{u^2+v^2+1}\right)$.

Since $\mathbf{R}^2$ is irreducible and since $f(\mathbf{R}^2)=\mathbf{S}^2\setminus\{(0,0,1)\}$, which is dense in $\mathbf{S}^2$, the variety $\mathbf{S}^2$ is irreducible. Furthermore, the rational maps $f:\mathbf{R}^2\to\mathbf{S}^2$ and $g:\mathbf{S}^2\to\mathbf{R}^2$ are rational inverses, so $\mathbf{S}^2\sim\mathbf{R}^2$.

6.15 Example: When $\mathbf{F}=\mathbf{R}$ we can use the birational maps of the above two examples to describe all the rational points on each of the varieties that were considered. For example, the birational map $f:\mathbf{R}\to V(y^2-x^3-x^2)\subseteq\mathbf{R}^3$ given by $f(t)=\big(t^2-1,t(t^2-1)\big)$ sends the rational number $\frac{a}{b}\in\mathbf{Q}\subseteq\mathbf{R}$ to the rational point $f\left(\frac{a}{b}\right)=\left(\frac{a^2-b^2}{b^2},\frac{a(a^2-b^2)}{b^3}\right)$, so all the rational points on the alpha curve $y^2=x^3+x^2$ are of this form where $a,b\in\mathbf{Z}$ with $b\neq 0$.

Similarly, the birational map $f:\mathbf{R}\to V(x^2+y^2-1)\subseteq\mathbf{R}^2$ given by $f(t)=\left(\frac{2t}{t^2+1},\frac{t^2-1}{t^2+1}\right)$ sends $\frac{a}{b}\in\mathbf{Q}$ to $f\left(\frac{a}{b}\right)=\left(\frac{2ab}{a^2+b^2},\frac{a^2-b^2}{a^2+b^2}\right)$, so all rational points on the unit circle $x^2+y^2=1$, except the point $(0,1)$, are of this form where $a,b\in\mathbf{Z}$ with $b\neq 0$.

6.16 Example: When $\mathbf{F}$ is a finite field, we can use birational maps to help count the number of points in a variety in $\mathbf{F}^n$. For example, let us count the number of points in $X = V(y^2 - x^3 - x^2) \subseteq \mathbf{F}^2$ where $\mathbf{F}$ is the finite field with $n = p^k$ elements.

Solution: Let $\mathbf{K}$ be the algebraic closure of $\mathbf{F}$. Then $X = V(y^2 - x^3 - x^2) \subseteq \mathbf{K}^2$ is irreducible, and the rational map $f : \mathbf{K} \to X$ given by $f(t) = \big(t^2 - 1, t(t^2 - 1)\big)$ determines a surjective map $f : \mathbf{F} \to X \cap \mathbf{F}$ which is 1:1 except that $f(1) = f(-1) = (0,0)$ (when $p = 2$, f is bijective). Thus when $p = 2$ there are n points in $X \cap \mathbf{F}^2$ and when $p \neq 2$ there are $n - 1$ points in $X \cap \mathbf{F}^2$.

6.17 Example: The **graph of a rational map** $f : X \to Y$ with domain U is defined to be the closure in $X \times Y$ of the graph of the map $f : U \to Y$, that is the closure of $\{(x, f(x)) | x \in U\}$. The natural **lift** $l : X \to \text{graph}\,(f)$ given by $l(x) = (x, f(x))$ and the natural **projection** $p : \text{graph}\,(f) \to X$ given by $p(x, y) = x$ are rational inverses, so $\text{graph}\,(f) \sim X$.

6.18 Definition: Let $a \in \mathbf{F}^n$. A **blow-up** of $\mathbf{F}^n$ at a is the graph $\widetilde{\mathbf{F}^n}$ of a projection g in $\mathbf{F}^n$ from the point a. The natural projection and lift $p : \widetilde{\mathbf{F}^n} \to \mathbf{F}^n$ and $l : \mathbf{F}^n \to \widetilde{\mathbf{F}^n}$ are rational inverses, so $\widetilde{\mathbf{F}^n} \sim \mathbf{F}^n$. If $X \subseteq \mathbf{F}^n$ is an irreducible variety with $a \in X$, then the corresponding **blow-up** of X, denoted by $\widetilde{X}$, is the graph of the restriction of g to X, or equivalently, the closure of $l\big(X \setminus \{a\}\big)$ in $\widetilde{\mathbf{F}^n}$.

6.19 Example: Find the blow-up $\widetilde{\mathbf{F}^2}$ and the natural projection and lift obtained from the projection in $\mathbf{F}^2$ from the origin to the line $x = 1$. Then find the induced blow-up of each of the two varieties $V(y^2 - x^3)$ and $V(y^2 - x^3 - x^2)$.

Solution: The projection in $\mathbf{F}^2$ from $(0,0)$ to the line $x = 1$, which we identify with $\mathbf{F}$, is the rational map $g : \mathbf{F}^2 \to \mathbf{F}$ given by $g(x, y) = \frac{y}{x}$, with domain $U = \{(x, y) | x \neq 0\}$. The graph of $g : U \to \mathbf{F}$ is the set $\{(x, y, z) | x \neq 0, z = y/x\} = \{(x, y, z) | x \neq 0, y - xz = 0\}$, and the corresponding blow-up of $\mathbf{F}^2$ at the origin is the closure of this set, $\widetilde{\mathbf{F}^n} = V(y - xz) \subseteq \mathbf{F}^3$. The natural projection and lift $p : \widetilde{\mathbf{F}^n} \to \mathbf{F}^2$ and $l : \mathbf{F}^2 \to \widetilde{\mathbf{F}^n}$ are given by $p(x, y, z) = (x, y)$ and $l(x, y) = \big(x, y, \frac{y}{x}\big)$.

Let $X = V(y^2 - x^3)$, and let $U = X \setminus \{(0,0)\}$. The points on X are of the form (t^2, t^3), so $l(U) = \{(t^2, t^3, t) | t \neq 0\}$, and $\widetilde{X} = \overline{l(U)} = \{(t^2, t^3, t) | t \in \mathbf{F}\} = V(x - z^2, y - z^3) \subseteq \mathbf{F}^3$. Thus $\widetilde{X}$ is the twisted cubic.

It is also worth considering the inverse image $p^{-1}(X)$ where $p(x, y, z) = (x, y)$. Verify that $p^{-1}(X) = V(y^2 - x^3, y - xz) = \widetilde{X} \cap V(x, y)$.

Now, let X be the alpha curve $X = V(y^2 - x^3 - x^2)$, and let $U = X \setminus \{(0,0)\}$. The points on X are of the form $\big(t^2 - 1, t(t^2 - 1)\big)$, so we have $l(U) = \big\{\big(t^2 - 1, t(t^2 - 1), t\big) \big| t \neq 0\big\}$ and $\widetilde{X} = \overline{l(U)} = \big\{\big(t^2 - 1, t(t^2 - 1), t\big) \big| t \in \mathbf{F}\big\} = V(x - z^2 + 1, y - z^3 + z^2)$.

6.20 Remark: Each of the two varieties X in the above example had a singularity at $(0,0)$, but the blow-up $\widetilde{X}$ was non-singular at $(0,0)$. It can be shown that by performing repeated blow-ups on a curve X at singular points, one eventually obtains a non-singular curve, called the **desingularization** of X, which is birational to X.

6.21 Definition: Let X and Y be irreducible affine varieties. Given a dominant rational map $f: X \to Y$, we define the **pullback** of f to be the map $f^*: K(Y) \to K(X)$ given by $f^*(g) = g \circ f$. Note that the composite $g \circ f$ is defined for any g because f is dominant. It is straightforward to check that $f^*: K(Y) \to K(X)$ is an $\mathbf{F}$-algebra homomorphism.

6.22 Theorem: *Let X and Y be irreducible affine varieties.*
(1) The map $f \mapsto f^*$ gives a bijective correspondence between the set of all dominant rational maps $f: X \to Y$ and the set of all $\mathbf{F}$-algebra homomorphisms $\phi: K(X) \to K(Y)$.
(2) If $f: X \to X$ is the identity then $f^*: K(X) \to K(X)$ is the identity, and if $f: X \to Y$ and $g: Y \to Z$ then $(g \circ f)^* = f^* \circ g^*$.
(3) $X \sim Y \iff K(X) \cong K(Y)$.

Proof: Let $\phi: K(Y) \to K(X)$ be any $\mathbf{F}$-algebra homomorphism. We wish to construct a dominant rational map f such that $f^* = \phi$. If such a map f exists then we must have $f^*(h) = \phi(h)$ for all $h \in K(Y)$, that is $h \circ f = \phi(h)$ for all $h \in K(Y)$. In particular, if f is given by $f = (f_1, \cdots, f_m)$ then we must have $f_i = y_i \circ f = \phi(y_i)$ for $i = 1, \cdots, m$. And indeed, if we define f by $f = \big(\phi(y_1), \cdots, \phi(y_n)\big)$ then for any $g = \frac{p}{q} = \frac{\sum a_{i_1,\cdots,i_m} {y_1}^{i_1} \cdots {y_m}^{i_m}}{\sum b_{j_1,\cdots,j_m} {y_1}^{j_1} \cdots {y_m}^{j_m}}$ in $K(Y)$ we have $f^*(g) = g \circ f = \frac{\sum a_{i_1,\cdots,i_m} \phi(y_1)^{i_1} \cdots \phi(y_m)^{i_m}}{\sum b_{j_1,\cdots,j_m} \phi(y_1)^{j_1} \cdots \phi(y_m)^{j_m}} = \phi(g)$ so that $f^* = \phi$.

This determines f uniquely as a rational map $f: X \to \mathbf{F}^m$ (and shows that $f \mapsto f^*$ is 1:1). We must show that the range of f is contained in Y and is dense.

Let U be the domain of f. Let $a \in U$. Let $g \in I(Y)$. Then $g = 0 \in K(Y)$. Since ϕ is an $\mathbf{F}$-algebra homomorphism, so that $\phi(0) = 0$, we have $\phi(g) = 0 \in K(X)$. Since $\phi(g) = f^*(g) = g \circ f$, we have $g \circ f = 0 \in K(X)$, so $g(f(x)) = 0$ for all $x \in U$. In particular, $g(f(a)) = 0$. Since $g \in I(Y)$ was arbitrary, we have $g(f(a)) = 0$ for all $g \in I(Y)$, that is $f(a) \in V(I(Y)) = Y$. Since $a \in U$ was arbitrary, the range of f is contained in Y.

It remains only to show that $f(U)$ is dense in Y, that is $\overline{f(U)} = Y$. We showed above that $f(U) \subseteq Y$ so we have $\overline{f(U)} \subseteq Y$, and we need to show that $Y \subseteq \overline{f(U)} = V(I(f(U)))$. Let $b \in Y$. Let $g \in I(f(U))$. Then $g(y) = 0$ for all $y \in f(U)$, so $g(f(x)) = 0$ for all $x \in U$. Writing $f = \frac{p}{q}$, where $p \in \mathbf{F}[x_1, \cdots, x_n]^m$ and $q \in \mathbf{F}[x_1, \cdots, x_n]$ with $q \notin I(X)$, we have $0 = g\big(f(x)\big) = g\big(\frac{p(x)}{q(x)}\big) q(x)^{\deg g}$ for all $x \in U \cap U_q$. Thus $(g \circ f) q^{\deg g} \in I(U \cap U_q) = I(X)$, and so $(g \circ f) q^{\deg g} = 0 \in K(X)$, hence $g \circ f = 0 \in K(X)$. Since $g \circ f = f^* \phi = \phi(g)$ we have $\phi(g) = 0 \in K(X)$. Since ϕ is a homomorphism of fields, it must be injective, and so $g = 0 \in K(Y)$. This means that $g(y) = 0$ for all y in the domain of g, but g is a polynomial, so $g(y) = 0$ for all $y \in Y$. In particular, $g(b) = 0$. Since $g \in I(f(U))$ was arbitrary, we have $g(b) = 0$ for all $g \in I(f(U))$ and so $b \in V(I(f(U))) = \overline{f(U)}$. Since $b \in Y$ was arbitrary, we have $Y \subseteq \overline{f(U)}$ as required.

Part (2) of the theorem is easy, and part (3) follows from parts (1) and (2).