5. Polynomial Maps and Isomorphism

5.1 Definition: Let X be a variety in $\mathbf{F}^n$. A polynomial $f \in \mathbf{F}[x_1, \cdots, x_n]$ determines a function $f : X \to \mathbf{F}$, and such a function is called a **polynomial map** from X to $\mathbf{F}$. The set of all polynomial maps from X to $\mathbf{F}$, denoted by $A(X)$, is an $\mathbf{F}$-algebra (meaning that $A(X)$ is both a commutative ring and a vector space over $\mathbf{F}$ and ring multiplication commutes with scalar multiplication) called the **ring of polynomial maps** on X, or simply the **coordinate ring** of X. Notice that for $f, g \in \mathbf{F}[x_1, \cdots, x_n]$, we have $f = g \in A(X)$ if and only if $f(x) = g(x)$ for all $x \in X$ if and only if $f - g \in I(X)$, so we can make the identification

$$A(X) = \mathbf{F}[x_1, \cdots, x_n] \big/ I(X)\,.$$

5.2 Note: X is irreducible $\iff$ $I(X)$ is prime $\iff$ $A(X)$ is an integral domain.

5.3 Note: Let $X \subseteq \mathbf{F}^n$ be an irreducible variety so that $A(X)$ is an integral domain. Let $\phi : \mathbf{F}[x_1, \cdots, x_n] \to A(X)$ be the natural map, and let $u_i = \phi(x_i) \in A(X)$. Then $A(X) = \mathbf{F}[u_1, \cdots, u_n]$.

5.4 Example: If $\mathbf{F}$ is infinite, then $I(\mathbf{F}^n) = \{0\}$ so we have $A(\mathbf{F}^n) = \mathbf{F}[x_1, \cdots, x_n]$, and so we can identify polynomial maps on $\mathbf{F}^n$ with polynomials in $\mathbf{F}[x_1, \cdots, x_n]$.

On the other hand, if $\mathbf{F}$ is finite, then there are only finitely many functions $f : \mathbf{F}^n \to \mathbf{F}$ (indeed if $|\mathbf{F}| = m$ then there are m^{m^n} such functions), so $A(\mathbf{F}^n)$ is finite, but there are infinitely many polynomials.

5.5 Example: Show that if $\mathbf{F}$ is finite then $A(\mathbf{F}^n)$ is the set of *all* functions $f : \mathbf{F}^n \to \mathbf{F}$.

Solution: For each $a \in \mathbf{F}$, define $g_a \in \mathbf{F}[x]$ by $g_a(x) = \dfrac{\prod_{b \in \mathbf{F}} (x - b)}{(x - a)}$. Note that $g_a(x) = 0$ for all $x \neq a$ but $g_a(a) \neq 0$. Define $\delta_a \in \mathbf{F}[x]$ by $\delta_a(x) = \frac{1}{g_a(a)}\, g_a(x)$. Then $\delta_a(x) = 0$ for all $x \neq a$ and $\delta_a(a) = 1$. Now, for each $a = (a_1, \cdots, a_n) \in \mathbf{F}^n$ define $\delta_a \in \mathbf{F}[x_1, \cdots, x_n]$ by $\delta_a(x) = \prod_{i=1}^{n} \delta_{a_i}(x_i)$. Then again we have $\delta_a(x) = 0$ for all $x \neq a$ and $\delta_a(a) = 1$. Finally, given any function $f : \mathbf{F}^n \to \mathbf{F}$ we have $f(x) = \sum_{a \in \mathbf{F}^n} f(a)\delta_a(x)$, so $f \in A(\mathbf{F}^n)$.

5.6 Definition: Let $X \subseteq \mathbf{F}^n$ and $Y \subseteq \mathbf{F}^m$ be varieties. An element $f \in \mathbf{F}[x_1, \cdots, x_n]^m$, that is an m-tuple $f = (f_1, \cdots, f_m)$ with each $f_i \in \mathbf{F}[x_1, \cdots, x_n]$, determines a map $f : X \to \mathbf{F}^m$, and if $f(X) \subseteq Y$ so that we have $f : X \to Y$, then we say that f is a **polynomial map** from X to Y.

5.7 Note: If $f : X \to Y$ is a polynomial map of affine varieties and if $\mathbf{Z} \subseteq Y$ is closed, then the inverse image $f^{-1}(Z)$ is closed in X. In other words, f is continuous. On the other hand, the image $f(X)$ need not be closed in Y.

Proof: Let us say $X \subseteq \mathbf{F}^n$ and $Z = V(S) \subseteq Y \subseteq \mathbf{F}^m$ where $S \subseteq \mathbf{F}[y_1, \cdots, y_m]$ and that the map $f : X \to Y$ is determined by $f \in \mathbf{F}[x_1, \cdots, x_n]^m$. Then we have

$$\begin{aligned} f^{-1}(Z) &= \{x \in X | f(x) \in Z\} \\ &= \{x \in X | g(f(x)) = 0 \text{ for all } g \in S\} \\ &= X \cap V(T) \text{ , where } T = \{g \circ f | g \in S\} \, . \end{aligned}$$

This shows that $f^{-1}(Z)$ is a subvariety of X.

To show that $f(X)$ need not be a subvariety of Y, let $X = V(xy - 1) \subseteq \mathbf{C}^2$ and let $Y = \mathbf{C}$ and let $f : X \to Y$ be the map $f(x, y) = x$. Then $f(X) = \{x \in \mathbf{C} | x \neq 0\}$, which is not a variety.

5.8 Example: If $f = (f_1, \cdots, f_m) \in \mathbf{F}[x_1, \cdots, x_n]^m$ then f defines a polynomial map $f : \mathbf{F}^n \to \mathbf{F}^m$ and we have $f^{-1}(0) = V(f_1, \cdots, f_m) \subseteq \mathbf{F}^n$.

5.9 Definition: Let X and Y be affine varieties. An **isomorphism** from X to Y is a bijective polynomial map $f : X \to Y$ the inverse of which is also a polynomial map. We say that X and Y are **isomorphic**, and we write $X \cong Y$, if there exists an isomorphism from X to Y.

5.10 Example: Let $X = V(y - x^2) \subseteq \mathbf{F}^2$. Show that $X \cong \mathbf{F}$.

Solution: The polynomial maps $f : \mathbf{F} \to X$ and $g : X \to \mathbf{F}$ given by $f(t) = (t, t^2)$ and $g(x, y) = x$ are easily seen to be inverses.

5.11 Example: Let X be the twisted cubic $X = V(y - x^2, z - x^3) \subseteq \mathbf{F}^3$. Show that $X \cong \mathbf{F}$.

Solution: The maps $f : \mathbf{F} \to X$ and $g : X \to \mathbf{F}$ given by $f(t) = (t, t^2, t^3)$ and $g(x, y, z) = x$ are inverses.

5.12 Definition: Let $X \subseteq \mathbf{F}^n$ be a variety and let $f : X \to \mathbf{F}^m$ be a polynomial map. The **graph** of f is

$$\text{graph}\,(f) = \Big\{(x, f(x)) \Big| x \in X\Big\} \subseteq X \times \mathbf{F}^m \subseteq \mathbf{F}^{n+m} \, .$$

5.13 Note: The graph of a polynomial map $f : X \to \mathbf{F}^n$ is a variety and $\text{graph}\,(f) \cong X$.

Proof: Say $X = V(S)$ where $S \subseteq \mathbf{F}[x_1, \cdots, x_n]$, and say f is given by $f = (f_1, \cdots, f_m) \in \mathbf{F}[x_1, \cdots, x_n]^m$. Then we have $\text{graph}\,(f) = V(T) \subseteq \mathbf{F}^{n+m}$ where $T = S \cup \{y - f(x)\} = S \cup \{y_1 - f_1(x), \cdots, y_m - f_m(x)\} \subseteq \mathbf{F}[x_1, \cdots, x_n, y_1, \cdots, y_m]$, so $\text{graph}\,(f)$ is a variety. And the maps $g : X \to \text{graph}\,(f)$ and $h : \text{graph}\,(f) \to X$ given by $g(x) = (x, f(x))$ and $h(x, y) = x$ are inverses, so $X \cong \text{graph}\,(f)$.

5.14 Definition: Let X and Y be affine varieties, and let $f : X \to Y$ be a polynomial map. Define the **pullback** of f to be the map $f^* : A(Y) \to A(X)$ given by $f^*(g) = g \circ f$. Note that f^* is an $\mathbf{F}$-algebra homomorphism since $f^*(g + h) = (g + h) \circ f = g \circ f + h \circ f$ and $f^*(gh) = (gh) \circ f = (g \circ f)(h \circ f)$ and $f^*(cg) = (cg) \circ f = c(g \circ f)$ for all $g, h \in A(Y)$ and $c \in \mathbf{F}$.

5.15 Theorem: *Let X and Y be affine varieties.*
(1) The map $f \mapsto f^*$ gives a bijective correspondence between the set of polynomial maps from X to Y and the set of $\mathbf{F}$-algebra homomorphisms from $A(Y)$ to $A(X)$.
(2) If $f : X \to X$ is the identity then $f^* : A(X) \to A(X)$ is the identity, and if $f : X \to Y$ and $g : Y \to Z$ then $(g \circ f)^* = f^* \circ g^*$.
(3) $X \cong Y$ if and only if $A(Y) \cong A(X)$.

Proof: Say $X \subseteq \mathbf{F}^n$ and $Y \subseteq \mathbf{F}^m$. Let $\phi : A(Y) \to A(X)$ be an $\mathbf{F}$-algebra homomorphism. We must show that there exists a unique polynomial map $f : X \to Y$ with $f^* = \phi$.

If such a polynomial map f exists, then we must have $f^*(h) = \phi(h)$ for all $h \in A(Y)$, that is $h \circ f = \phi(h)$ for all $h \in A(Y)$. In particular, we must have $f_i = y_i \circ f = \phi(y_i)$ for each $i = 1, \cdots m$, where $y_i \in A(Y)$. This shows that the map f, if it exists, is unique, and must be given by $f = \big(\phi(y_1), \cdots, \phi(y_m)\big)$.

Now, choose polynomials $f_i \in \mathbf{F}[x_1, \cdots, x_n]$ so that $\phi(y_i) = f_i \in A(X)$, and then set $f = (f_1, \cdots, f_m) \in \mathbf{F}[x_1, \cdots, x_n]^m$. Then f defines a polynomial map $f : X \to \mathbf{F}^m$. We shall show that $f(X) \subseteq Y$ so that $f : X \to Y$ and that $f^* = \phi$.

Since ϕ is an $\mathbf{F}$-algebra homomorphism, for any $g = \sum c_{k_1,\cdots,k_m} {y_1}^{k_1} \cdots {y_m}^{k_m}$ in $\mathbf{F}[y_1, \cdots, y_m]$ we have $\phi(g) = \sum c_{k_1,\cdots,k_m} \phi(y_1)^{k_1} \cdots \phi(y_m)^{k_m} = \sum c_{k_1,\cdots,k_m} {f_1}^{k_1} \cdots {f_m}^{k_m} = g \circ f = f^*(g) \in A(X)$, thus $\phi = f^*$.

It remains to show that we have $f(X) \subseteq Y$. Let $x \in X$. Then for any $g \in I(Y)$ we have $g = 0 \in A(Y)$ so $g \circ f = f^* g = \phi(g) = \phi(0) = 0 \in A(X)$, and so $g(f(x)) = 0$. Since $g(f(x)) = 0$ for all $g \in I(Y)$, we have $f(x) \in V(I(Y)) = Y$. This shows that indeed $f(X) \subseteq Y$ so $f : X \to Y$, and completes the proof of part (1).

Part (2) is easy, and part (3) follows from parts (1) and (2).

5.16 Corollary: *Let X and Y be affine varieties. If $X \cong Y$ then X is irreducible if and only if Y is irreducible.*

Proof: If $X \cong Y$ then we have $A(X) \cong A(Y)$ so X is irreducible $\iff$ $I(X)$ is prime $\iff$ $A(X)$ is an integral domain $\iff$ $A(Y)$ is an integral domain $\iff$ $I(Y)$ is prime $\iff$ Y is irreducible.

5.17 Example: Let $\mathbf{F}$ be infinite. Show that the twisted cubic $X = V(y - x^2, z - x_3) \subseteq \mathbf{F}^2$ is irreducible.

Solution: In example 5.11 we saw that $X \cong \mathbf{F}$. Since $\mathbf{F}$ is infinite, $\mathbf{F}$ is irreducible, and so X is irreducible, too.

5.18 Example: Let $\mathbf{F}$ be an infinite field. Find the irreducible components of the variety $X = V(x^2 - yz, xz - x) \subseteq \mathbf{F}^3$.

Solution: Let $(x, y, z) \in X$. We have $xz - x = 0$ so $x(z - 1) = 0$ so $x = 0$ or $z = 1$. We also have $x^2 = yz$, so when $x = 0$ we have $yz = 0$ so that $y = 0$ or $z = 0$, and when $z = 1$ we have $x^2 = y$. Thus $X = V(x, y) \cup V(x, z) \cup V(z - 1, y - x^2)$, a union of two lines and a parabola in $\mathbf{F}^3$. Since $V(x, y) \equiv \mathbf{F}$ and $V(x, z) \equiv \mathbf{F}$ and $V(z - 1, y - x^2) \equiv V(y - x^2) \subseteq \mathbf{F}^2$, these three varieties are all irreducible, so they are the irreducible components of X.

5.19 Definition: Let X and Y be affine varieties and let $f : X \to Y$ be a polynomial map. We say that f is **dominant** when $f(X)$ is **dense** in Y, that is when $\overline{f(X)} = Y$, and we say that f has a **left polynomial inverse** when there exists a polynomial map $g \in \mathbf{F}[y_1, \cdots, y_m]^n$ such that $g\big(f(x)\big) = x$ for all $x \in X$.

5.20 Theorem: *Let $f : X \to Y$ be a polynomial map between affine varieties, so we have $f^* : A(Y) \to A(X)$. Then*

(1) f^* is injective if and only if f is dominant, and
(2) f^* is surjective if and only if f has a left polynomial inverse.

Proof: Suppose that f^* is injective. Then for $g \in \mathbf{F}[x_1, \cdots, x_n]$ we have $g \in I(f(X)) \iff g(f(x)) = 0$ for all $x \in X \iff \phi(g) = 0 \in AX \iff g = 0 \in A(Y) \iff g \in I(Y)$, so $I(f(X)) = I(Y)$ and hence $\overline{f(X)} = V(I(f(X))) = V(I(Y)) = Y$.

Conversely, suppose that f is dominant. Then for $g \in A(Y)$ given by $g \in \mathbf{F}[y_1, \cdots, y_m]$ we have $\phi(g) = 0 \in A(X) \iff g(f(x)) = 0$ for all $x \in X \iff g(y) = 0$ for all $y \in f(X) \iff g \in I(f(X)) = I\big(\overline{f(X)}\big) = I(Y) \iff g = 0 \in A(Y)$, so ϕ is injective.

Now suppose that f^* is surjective. For each i, choose $g_i \in \mathbf{F}[y_1, \cdots, y_m]$ so that $f^*(g_i) = x_i \in A(X)$, that is $g_i(f(x)) = x_i$ for all $x \in X$. Let $g = (g_1, \cdots, g_n)$. Then $g(f(x)) = x$ for all $x \in X$, so g is a left polynomial inverse for f.

Suppose, conversely, that f has a left polynomial inverse, say $g \in \mathbf{F}[y_1, \cdots, y_m]^n$. Given $h \in A(X)$, extend h to $h \in \mathbf{F}[x_1, \cdots, x_n]$ and set $k = h \circ g \in \mathbf{F}[y_1, \cdots, y_m]^n$. Then for $k \in A(Y)$, we have $f^*(k) = k \circ f = h \circ g \circ f = h \in A(X)$. Thus f^* is surjective.

5.21 Example: Let $\mathbf{F}$ be an infinite field. Let $X = M_{\leq 1}(2, \mathbf{F}) = V(x_1x_4 - x_2x_3) \subseteq \mathbf{F}^4$. Show that X is irreducible.

Solution: Define $f : \mathbf{F}^3 \to X$ by $f(t_1, t_2, t_3) = \begin{pmatrix} t_1 & t_2 \\ t_1t_3 & t_2t_3 \end{pmatrix}$. We claim that f is dominant. Let $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in X$. If $x \in f(\mathbf{F}^3)$ then of course $x \in \overline{f(\mathbf{F}^3)}$, so suppose $x \notin f(\mathbf{F}^3)$. Notice that this implies that the first row of x is zero and the second row of x is nonzero. Let $g \in I(f(\mathbf{F}^3))$ so that $g\big(f(t_1, t_2, t_3)\big) = 0$ for all t_1, t_2, t_3. Then in particular, $g\big(f\big(\epsilon\, x_3, \epsilon\, x_4, \frac{1}{\epsilon}\big)\big) = g\begin{pmatrix} \epsilon x_3 & \epsilon x_4 \\ x_3 & x_4 \end{pmatrix} = 0$ for all $\epsilon \neq 0$. Since $\mathbf{F}$ is infinite, this implies that $g\begin{pmatrix} \epsilon x_3 & \epsilon x_4 \\ x_3 & x_4 \end{pmatrix} = 0 \in \mathbf{F}[\epsilon]$ and in particular that $g(x) = g\begin{pmatrix} 0 & 0 \\ x_3 & x_4 \end{pmatrix} = 0$. Since $g \in I(f(\mathbf{F}^3))$ was arbitrary, we have $x \in V(I(f(\mathbf{F}^3))) = \overline{f(\mathbf{F}^3)}$. Thus f is dominant.

Since f is dominant, $f^* : A(X) \to A(\mathbf{F}^3)$ is injective, and since $A(\mathbf{F}^3) = \mathbf{F}[t_1, t_2, t_3]$ is an integral domain, $A(X)$ must also be an integral domain. So $I(X)$ is prime, and X is irreducible.

5.22 Example: Let $\mathbf{F}$ be infinite and let $X = V(y^2 - x^3) \subseteq \mathbf{F}^2$. Prove that $X \not\cong \mathbf{F}$.

Solution: The map $f : \mathbf{F} \to X$ given by $f(t) = (t^2, t^3)$ is surjective and hence dominant, so it induces an inclusion $f^* : A(X) \to \mathbf{F}[t]$ which is given by $f^*(g(x, y)) = g(t^2, t^3)$. Thus $A(X) \cong f^*(A(X)) = \big\{g(t^2, t^3) \in \mathbf{F}[t] \big| g \in \mathbf{F}[x, y]\big\}$. Notice that the elements $x, y \in A(X)$ are both irreducible since $f^*(x) = t^2$ and $f^*(y) = t^3$ are both irreducible in $f^*(A(X))$. But we have $x^3 = y^2 \in A(X)$, so $A(X)$ is not a unique factorization domain, and hence we cannot have $A(X) \cong \mathbf{F}[t]$. Thus $X \not\cong \mathbf{F}$.