4. Affine Maps and Affine Equivalence

4.1 Definition: Recall again that an **affine space** in $\mathbf{F}^n$ is a set of the form

$$p + V = \{p + v \big| v \in V\}$$

for some $p \in \mathbf{F}^n$ and some vector space V in $\mathbf{F}^n$. Verify that for any points $p, q \in \mathbf{F}^n$ and for any vector space $V, W \subseteq \mathbf{F}^n$, we have $p+V = q+W \iff \big(q \in p+V \text{ and } V = W\big)$. In particular, the vector space V is uniquely determined by the affine space $p+V$, so it makes sense to call V the **associated vector space** of the affine space $p+V$, and to define the **dimension** of the affine space $X = p+V$, denoted by $\dim(X)$, to be the dimension of the vector space V over the field $\mathbf{F}$. An affine space of dimension 0 is called a **point**, an affine space of dimension 1 is a **line**, and an affine space of dimension 2 is a **plane**.

4.2 Definition: An **affine map** from $\mathbf{F}^n$ to $\mathbf{F}^m$ is a function $f : \mathbf{F}^n \to \mathbf{F}^m$ of the form

$$f(x) = Ax + b$$

for some $m \times n$ matrix A with entries in $\mathbf{F}$ and some vector $b \in \mathbf{F}^m$. Notice that the matrix A and the vector b are uniquely determined from f; indeed $b = f(0)$ and A is determined by $Ax = f(x) - f(0)$ for all x $\big($so A is the matrix with columns $f(e_i) - f(0)$, where the e_i are the standard basis vectors in $\mathbf{F}^n\big)$. We call the matrix A the **associated matrix** of the affine map f, and we define the **rank** of f to be the rank of the matrix A. Notice that an affine map $f : \mathbf{F}^n \to \mathbf{F}^m$ is bijective if and only if $n = m$ and its associated matrix is invertible. An affine equivalence $f : \mathbf{F}^n \to \mathbf{F}^n$ is called an **affine change of coordinates**.

4.3 Definition: Let $X \subseteq \mathbf{F}^n$ and $Y \subseteq \mathbf{F}^m$ be varieties. If $f : \mathbf{F}^n \to \mathbf{F}^m$ is an affine map and if $f(X) \subseteq Y$ then f restricts to a map $f : X \to Y$, and such a map is called an **affine map** from X to Y. An **affine equivalence** from X to Y is a bijective affine map $f : X \to Y$ whose inverse $g : Y \to X$ is also affine. We say that X and Y are (affinely) **equivalent** and we write $X \equiv Y$, if there exists an affine equivalence from X to Y.

4.4 Example: Let $X \subseteq \mathbf{F}^n$ and $Y \subseteq \mathbf{F}^m$ be affine spaces. Show that $X \equiv Y \iff \dim(X) = \dim(Y)$.

Solution: Suppose that $X \equiv Y$. Let $f : \mathbf{F}^n \to \mathbf{F}^m$ be an affine map inducing an equivalence $f : X \to Y$, and say $f(x) = Ax + b$. Let V and W be the vector spaces associated to X and Y respectively. Define $g : \mathbf{F}^n \to \mathbf{F}^m$ by $g(x) = Ax$. Verify that the restriction of g to V gives a vector space isomorphism $g : V \to W$. Since V and W are isomorphic, they must have the same dimension. Thus $\dim(X) = \dim(Y)$.

Conversely, say $X = p + V \subseteq \mathbf{F}^n$ and $Y = q + W \subseteq \mathbf{F}^m$ and suppose that $\dim(V) = \dim(W) = r$. Choose an (ordered) basis $\{v_1, \cdots, v_r\}$ for V and extend it to a basis $\{v_1, \cdots, v_n\}$ for $\mathbf{F}^n$. Choose a basis $\{w_1, \cdots, w_r\}$ for W and extend it to a basis $\{w_1, \cdots, w_m\}$ for $\mathbf{F}^m$. Let A be any $m \times n$ matrix such that $Av_i = w_i$ for $i = 1, \cdots, r$ $\big($if P is the $n \times n$ matrix with columns v_i and if Q is any $m \times n$ matrix whose first r columns are the vectors w_i, then $A = Q\,P^{-1}$ will work$\big)$, and let B be any $n \times m$ matrix such that $Bw_i = v_i$ for $i = 1, \cdots, r$. Define $h : \mathbf{F}^n \to \mathbf{F}^m$ and $k : \mathbf{F}^m \to \mathbf{F}^n$ by $h(x) = Ax$ and $k(y) = By$. Then $h : V \to W$ is a vector space isomorphism with inverse $k : W \to V$. Define $f : \mathbf{F}^n \to \mathbf{F}^m$ and $g : \mathbf{F}^m \to \mathbf{F}^n$ by $f(x) = A(x - p) + q$ and $g(y) = B(y - q) + p$. Verify that $f : X \to Y$ is bijective with inverse $g : Y \to X$.

4.5 Example: By the diagonalizability of symmetric bilinear forms and by Sylvester's Law of Inertia, it can be shown that for any polynomial $f \in \mathbf{R}[x,y]$ of degree 2, the variety $V(f) \subseteq \mathbf{R}^2$ is equivalent to one of the following varieties: the circle $V(x^2+y^2-1)$, the hyperbola $V(x^2-y^2-1)$, the parabola $V(y-x^2)$, the pair of intersecting lines $V(x^2-y^2)$, the pair of parallel lines $V(y^2-1)$, the single line $V(y^2)=V(y)$, the single point $V(x^2+y^2) = \{(0,0)\}$ or the empty set $V(x^2+y^2+1) = V(y^2+1) = \emptyset$.

4.6 Example: Let $X = V(y-x^2) \subseteq \mathbf{R}^2$ and let $Y = V(u^2+v^2+2uv-2u-3v+1) \subseteq \mathbf{R}^2$. Find an affine equivalence $f : X \to Y$.

Solution: Note that $u^2+v^2+2uv-2u-3v+1 = (u+v-1)^2 - v$, and we have $(x,y) = (u+v-1, v)$ if and only if $(u,v) = (x-y+1, y)$, so the affine maps $f : X \to Y$ and $g : Y \to X$ given by $f(x,y) = (x-y+1, y)$ and $g(u,v) = (u+v-1, v)$ are inverses.

4.7 Example: It can be shown, again by by the diagonalizability of symmetrc bilinear forms, that for any $f \in \mathbf{C}[x,y]$ of degree 2 the variety $V(f) \subseteq \mathbf{C}^2$ is equivalent to one of the following varieties: the circle $V(x^2+y^2-1)$, the parabola $V(y-x^2)$, the pair of intersecting lines $V(x^2-y^2)$ and the single line $V(y^2) = V(y)$.

4.8 Example: Let $X = V(x^2+y^2-1) \subseteq \mathbf{C}^2$ and let $Y = V(u^2-v^2+1) \subseteq \mathbf{C}^2$. Find an affine equivalence $f : X \to Y$.

Solution: Note that $Y = V(-u^2+v^2-1)$ and $-u^2+v^2-1 = (iu)^2+v^2-1$, so we can define affine equivalence $f : X \to Y$ and its inverse $g : Y \to X$ by $f(x,y) = (ix, y)$ and $g(u,v) = (iu, v)$.