11. Projective Varieties

11.1 Definition: Let W be a vector space over a field $\mathbf{F}$. The **projectivization of** W is the set of all 1-dimensional subspaces of W. For $0 \neq x \in W$, write $[x] = \mathrm{Span}_{\mathbf{F}}\{x\}$. Then

$$\mathbf{P}(W) = \big\{[x]\big|0 \neq x \in V\big\}$$

The projectivization of the vector space $\mathbf{F}^{n+1}$ is called the **projective n-space** over $\mathbf{F}$, and denoted by $\mathbf{P}^n$ or $\mathbf{P}^n(\mathbf{F})$:

$$\mathbf{P}^n(\mathbf{F}) = \big\{[x]\big|0 \neq x \in \mathbf{F}^{n+1}\big\}.$$

$\mathbf{P}^1(\mathbf{F})$ is called the **projective line** and $\mathbf{P}^2(\mathbf{F})$ is called the **projective plane** over $\mathbf{F}$. An element of $\mathbf{P}^n$ (which is a line through the origin in $\mathbf{F}^{n+1}$) is called a **point** in $\mathbf{P}^n$. If V is a 2-dimensional subspace of W then the set $\mathbf{P}(V) \subseteq \mathbf{P}(W)$ is called a **line** in $\mathbf{P}(W)$. More generally, if V is a $(k+1)$-dimensional subspace of W, then $\mathbf{P}(V)$ is called a k-dimensional projective space in $\mathbf{P}(W)$. For each $k = 1, 2, \cdots, n+1$, let

$$U_k = \big\{[x_1, \cdots, x_{n+1}] \in \mathbf{P}^n \big| x_k \neq 0\big\}.$$

We can identify U_k with $\mathbf{F}^n$ using the k^{th} **inclusion map** $\phi_k : \mathbf{F}^n \to U_k \subseteq \mathbf{P}^n$ given by

$$\begin{aligned}\phi_k(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1}) &= [x_1, \cdots, x_{k-1}, 1, x_{k+1}, \cdots, x_{n+1}] \\ {\phi_k}^{-1}\big([x_1, \cdots, x_{n+1}]\big) &= \left(\tfrac{x_1}{x_k}, \cdots, \tfrac{x_{k-1}}{x_k}, \tfrac{x_{k+1}}{x_k}, \cdots, \tfrac{x_{n+1}}{x_k}\right)\end{aligned}$$

and we can identify $\mathbf{P}^n \setminus U_k$ with $\mathbf{P}^{n-1}$ using the bijection $\psi_k : \mathbf{P}^n \setminus U_k \to \mathbf{P}^{n-1}$ given by

$$\psi_k\big([x_1, \cdots, x_{k-1}, 0, x_{k+1}, \cdots, x_n]\big) = [x_1, \cdots, x_{k-1}, x_{k+1} \cdots, x_n].$$

For fixed k we sometimes write $\mathbf{P}^n = \mathbf{F}^n \cup \mathbf{P}^{n-1}$, where we have made the identifications $\mathbf{F}^n = U_k \subseteq \mathbf{P}^n$ and $\mathbf{P}^{n-1} = \mathbf{P}^n \setminus \mathbf{F}^k$, and we then refer to $\mathbf{P}^{n-1}$ as the **projective space at infinity**.

11.2 Example: There is only one 1-dimensional subspace of $\mathbf{F}$, namely $\mathbf{F}$ itself, so $\mathbf{P}^0(\mathbf{F})$ consists of a single point. Also, fixing $k = 1$ or 2, we can write $\mathbf{P}^1(\mathbf{F}) = \mathbf{F} \cup \{\infty\}$, where we have identified $\mathbf{F} = U_k$ and $\{\infty\} = \mathbf{P}^0 = \mathbf{P}^1 \setminus U_k$.

11.3 Example: The real projective line $\mathbf{P}^1(\mathbf{R})$ is the set of lines in $\mathbf{R}^2$ through the origin. U_1 is the set of all these lines except the y-axis, each of which intersects the line $x = 1$, and indeed it is natural to identify U_1 with the line $x = 1$. Similarly it is natural to identify U_2, which is the set of all the lines through the origin except the x-axis, with the line $y = 1$. Also, since each line through the origin intersects the unit circle $\mathbf{S}^1 = V(x^2+y^2-1)$ at two antipodal points, we can think of $\mathbf{P}^1(\mathbf{R})$ as a copy of $\mathbf{S}^1$ with antipodal points identified.

11.4 Example: The real projective plane $\mathbf{P}^2(\mathbf{R})$ is the set of lines through the origin in $\mathbf{R}^3$. The set U_1, which is the set of these lines which do not lie in the plane $x = 0$, can be identified with the plane $x = 1$, U_2 can be identified with the plane $y = 1$, and U_3 can be identified with the plane $z = 1$. We can also visualize $\mathbf{P}^2$ as a copy of the unit sphere $\mathbf{S}^2 = V(x^2 + y^2 + z^2 - 1)$ with antipodal points identified. The real projective plane is an example of a non-orientable surface. It was first discovered and studied by artists who were interested in perspective: if one's eye is at the origin in $\mathbf{R}^3$, then all the points along a given line through the origin will appear to be coincident and can be identified.

11.5 Example: The complex projective line $\mathbf{P}^1(\mathbf{C})$ can be thought of as $\mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$. It can also be identified with the unit sphere $\mathbf{S}^2$, and it is then referred to as the **Riemann sphere**. Under this identification, the maps ϕ_1 and ϕ_2 correspond to the stereographic projections from the north and south poles.

11.6 Example: Let $\mathbf{F}$ be the field with $n = p^k$ elements. There are $n^2 - 1$ non-zero points in $\mathbf{F}^2$, each of these points spans a 1-dimensional vector space in $\mathbf{F}^2$, and each 1-dimensional vector space contains $n-1$ non-zero points, so the number of 1-dimensional spaces in $\mathbf{F}^2$ is equal to $\frac{n^2-1}{n-1} = n+1$. In other words, the projective line $\mathbf{P}^1(\mathbf{F})$ consists of $n+1$ points.

11.7 Definition: A set $E \subseteq \mathbf{F}^{n+1}$ is called **homogeneous** if $E = \{0\}$ or E is a union of one-dimensional vector spaces in $\mathbf{F}^{n+1}$. Equivalently, E is homogeneous if for all $x \in E$ we have $tx \in E$ for all $t \in \mathbf{F}$, or again equivalently, if for all $0 \neq x \in E$ we have $[x] \subseteq E$. There is a natural correspondence between homogeneous sets in $\mathbf{F}^{n+1}$ and sets in $\mathbf{P}^n$: given a homogeneous set $E \subseteq \mathbf{F}^{n+1}$, the **projectivization** of E is the set

$$\mathbf{P}(E) = \big\{[x] \big| 0 \neq x \in E\big\} \subseteq \mathbf{P}^n$$

and given a set $F \subseteq \mathbf{P}^n$, the **affine cone** of F is the homogeneous set

$$\mathbf{A}(F) = \bigcup_{[x]\in F} [x] \,.$$

This correspondence is bijective except that $\mathbf{P}(\{0\}) = \mathbf{P}(\emptyset) = \emptyset$, while $\mathbf{A}(\emptyset) = \emptyset$. We define a **projective algebraic variety in $\mathbf{P}^n$**, or simply a **variety in $\mathbf{P}^n$**, to be the projectivization of a homogeneous variety in $\mathbf{F}^{n+1}$.

11.8 Definition: The **Zariski topology** on $\mathbf{P}^n$ is the topology in which the closed sets are the projective varieties. For a set $E \subseteq \mathbf{P}^n$ we write $\overline{E}$ for the **closure** of E, that is the smallest closed set in $\mathbf{P}^n$ which contains E.

11.9 Definition: A projective variety $X \subseteq \mathbf{P}^n$ is **irreducible** if $X \neq \emptyset$ and X is not equal to the union of any two proper subvarieties.

11.10 Note: If $X \subseteq \mathbf{F}^{n+1}$ is a homogeneous variety then the irreducible components of X are also homogeneous. Consequently, if $Y = \mathbf{P}(X) \subseteq \mathbf{P}^n$ is the projectivization of X, then Y is irreducible if and only if X is irreducible. Moreover, since X has a unique decomposition into irreducible components, Y also has a unique decomposition into irreducible components, and the irreducible components of Y are the projectivizations of the irreducible components of X.

Proof: The proof is left as an exercise.

11.11 Definition: A polynomial $f \in \mathbf{F}[x_1, \cdots, x_{n+1}]$ of degree k is called **homogeneous** if all the monomials $x_1^{i_1} \cdots x_n^{i_n}$ which occur in f (with non-zero coefficient) have the same total degree $k = i_1 + \cdots + i_n$. The zero polynomial is also considered to be homogeneous. Note that every polynomial $f \in \mathbf{F}[x_1, \cdots, x_{n+1}]$ of degree d can be expressed uniquely in the form $f = f_0 + f_1 + \cdots + f_d$ where each f_k is homogeneous of degree k. The polynomials f_k are called the **homogeneous components of f**.

11.12 Example: In $\mathbf{R}[x, y]$, $f(x, y) = 1+2x-3y+x^2-2xy+5y^2-x^3+4x^2y$ is a polynomial of degree 3, and we have $f = f_0 + f_1 + f_2 + f_3$ where $f_0(x, y) = 1$, $f_1(x, y) = 2x - 3y$, $f_2(x, y) = x^2 - 2xy + 5y^2$ and $f_3(x, y) = -x^3 + 4x^2y$ are the homogeneous components.

11.13 Definition: Note that if $f \in \mathbf{F}[x_1, \cdots, x_{n+1}]$ is homogeneous of degree k then we have $f(tx) = t^k f(x)$ for all $t \in \mathbf{F}$, $x \in \mathbf{F}^{n+1}$, so for $0 \neq x \in \mathbf{F}^{n+1}$ we have $f(x) = 0 \iff f(tx) = 0$ for all $t \in \mathbf{F}$, and so it makes sense to define

$$f\big([x]\big) = 0 \iff f(x) = 0\,.$$

Let $S \subseteq \mathbf{F}[x_1, \cdots, x_{n+1}]$ be a collection of homogeneous polynomials. Notice that the affine variety $V(S) \subseteq \mathbf{F}^{n+1}$ is homogeneous and that for $0 \neq x \in \mathbf{F}^{n+1}$ we have $x \in V(S) \subseteq \mathbf{F}^{n+1} \iff f(x) = 0$ for all $f \in S \iff f\big([x]\big) = 0$ for all $f \in S$. We define the **projective variety in $\mathbf{P}^n$ cut out by** S to be the set

$$V(S) = \big\{[x] \in \mathbf{P}^n \big| f\big([x]\big) = 0 \text{ for all } f \in S\big\} \subseteq \mathbf{P}^n\,,$$

so the projective variety $V(S) \subseteq \mathbf{P}^n$ is the projectivization of the homogeneous affine variety $V(S) \subseteq \mathbf{F}^{n+1}$.

11.14 Example: Let $f(x, y) = x^2 - 2xy - 3y^2 = (x - 3y)(x + y) \in \mathbf{F}[x, y]$. Note that f is homogeneous of degree 2. Then the homogeneous affine variety cut out by f is $V(f) = V(x - 3y) \cup V(x + y) = [3, 1] \cup [-1, 1] \subseteq \mathbf{F}^2$, and the projective variety cut out by f is $V(f) = \big\{[3, 1], [-1, 1]\big\} \subseteq \mathbf{P}^2$.

11.15 Example: Let $f, g \in \mathbf{F}[x, y, z]$ be the homogeneous polynomials $f(x, y, z) = x+y+z$ and $g(x, y, z) = x^2 + y^2 - z^2$. Then in $\mathbf{F}^3$ we have several homogeneous affine varieties: $V(f) \subseteq \mathbf{F}^3$, is a plane through the origin; if $\operatorname{char}\mathbf{F} \neq 2$ then $V(g) \subseteq \mathbf{F}^3$ is a double cone $\big($if $\operatorname{char}\mathbf{F} = 2$ then $V(g) = V\big((x + y + z)^2\big) = V(x + y + z) = V(f)\big)$; and we have $(x, y, z) \in V(f, g)$ when $z = -(x+y)$ and $0 = x^2 + y^2 - z^2 = x^2 + y^2 - (x + y)^2 = -2xy$ so $V(f, g) = V(x, z+y) \cup V(y, z+x) = [0, -1, 1] \cup [-1, 0, 1] \subseteq \mathbf{F}^3$. In $\mathbf{P}^2$ we have corresponding projective varieties: $V(f) \subseteq \mathbf{P}^2$ is a projective line, and $V(f, g) = \big\{[0, -1, 1], [-1, 0, 1]\big\}$.

11.16 Definition: An ideal $A \subseteq \mathbf{F}[x_1, \cdots, x_{n+1}]$ is called **homogeneous** if it is generated by some set of homogeneous polynomials. For a homogeneous ideal $A \subseteq \mathbf{F}[x_1, \cdots, x_{n+1}]$, we define the **projective variety of A in $\mathbf{P}^n$** to be

$$V(A) = \big\{[x] \in \mathbf{P}^n \big| f\big([x]\big) = 0 \text{ for all homogeneous } f \in A\big\} \subseteq \mathbf{P}^n\,.$$

11.17 Theorem: *Let $\mathbf{F}$ be an infinite field and let $E \subseteq \mathbf{F}^{n+1}$ be any homogeneous set. Then $I(E) \subseteq \mathbf{F}[x_1, \cdots, x_{n+1}]$ is a homogeneous ideal, which is generated by a finite set of homogeneous polynomials.*

Proof: Say $I(E) = \langle f_1, \cdots, f_l\rangle$. For each i, write $f_i = f_{i0} + f_{i1} + \cdots + f_{ld_i}$ with each f_{ij} homogeneous of degree j. We claim that $I(E) = \big\langle\{f_{ij}\}\big\rangle$. It is clear that $I(E) \subseteq \big\langle\{f_{ij}\}\big\rangle$ since for each i,$f_i = \sum f_{ij} \in \big\langle\{f_{ij}\}\big\rangle$, so we need to show that each $f_{ij} \in I(E)$. Fix i and let $a \in E$. Since E is homogeneous we have $ta \in E$ for all $t \in \mathbf{F}$, so

$$\begin{aligned}
&f_i(ta) = 0 \text{ for all } t\\
&f_{i0}(ta) + f_{i1}(ta) + \cdots + f_{id_i}(ta) = 0 \text{ for all } t\\
&f_{i0}(a) + f_{i1}(a)\,t + \cdots + f_{id_i}(a)\,t^{d_i} = 0 \text{ for all } t\,.
\end{aligned}$$

Since $\mathbf{F}$ is infinite, this implies that $f_{ij} = 0$ for all j. Thus $f_{ij} \in I(E)$ for all j.

11.18 Example: If $\mathbf{F}$ is the finite field with $n = p^k$ elements, then $\mathbf{F}$ is a homogeneous set but $I(\mathbf{F}) = \langle x^n - x\rangle$ is not a homogeneous ideal.

11.19 Definition: Given a homogeneous polynomial $f \in \mathbf{F}[x_1, \cdots, x_{n+1}]$, we define the k^{th} **dehomogenization** of f to be the polynomial $g_k(f) \in \mathbf{F}[x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1}]$ given by

$$g_k(f)(x_1, \cdots, x_{k-1}, x_{x+1}, \cdots, x_{n+1}) = f(x_1, \cdots, x_{k-1}, 1, x_{k+1}, \cdots, x_{n+1}) .$$

Given $g \in \mathbf{F}[x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1}]$ we define the k^{th} **homogenization of** g to be the homogeneous polynomial $h_k(g) \in \mathbf{F}[x_1, \cdots, x_{n+1}]$ given by

$$h_k(g)(x_1, \cdots, x_{n+1}) = g\left(\frac{x_1}{x_k}, \cdots, \frac{x_{k-1}}{x_k}, \frac{x_{k+1}}{x_k}, \cdots, \frac{x_{n+1}}{x_k}\right) {x_k}^{\deg g} .$$

11.20 Example: Let $f(x,y,z) = x^3 + 2x^2y - xyz \in \mathbf{F}[x,y,z]$. Note that f is homogeneous. The dehomogenizations $f_k = g_k(f)$ are given by

$$\begin{aligned} f_1(y,z) &= f(1,y,z) = 1 + 2y - yz \\ f_2(x,z) &= f(x,1,z) = x^3 + 2x^2 - xz \\ f_3(x,y) &= f(x,y,1) = x^3 + 2x^2y - xy . \end{aligned}$$

The homogenizations $h_k = h_k(f_k)$ are given by

$$\begin{aligned} h_1(y,z) &= f_1\big(\tfrac{y}{x}, \tfrac{z}{x}\big)x^2 = \big(1 + 2\tfrac{y}{x} - \tfrac{yz}{x^2}\big)x^2 = x^2 + 2xy - yz \\ h_2(x,z) &= f_2\big(\tfrac{x}{y}, \tfrac{z}{y}\big)y^3 = \big(\tfrac{x^3}{y^3} + 2\tfrac{x^2}{y^2} - \tfrac{xz}{y^2}\big)y^3 = x^3 + 2x^2y - xyz \\ h_3(x,y) &= f_3\big(\tfrac{x}{z}, \tfrac{y}{z}\big)z^3 = \big(\tfrac{x^3}{z^3} + 2\tfrac{x^2y}{z^3} - \tfrac{xy}{z^2}\big)z^3 = x^3 + 2x^2y - xyz . \end{aligned}$$

Notice that $f = h_2 = h_3 = xh_1$.

11.21 Remark: Recall that by using the k^{th} inclusion map $\phi_k : \mathbf{F}^n \to U_k \subseteq \mathbf{P}^n$, we can consider $\mathbf{F}^n$ to be a subset of $\mathbf{P}^n$. When we do this we can restrict a given projective variety $X \subseteq \mathbf{P}^n$ to an affine variety $X_k \subseteq \mathbf{F}^n$ and, on the other hand, we can extend a given affine variety $X \subseteq \mathbf{F}^n$ to a projective variety $\overline{X} \subseteq \mathbf{P}^n$. This is described more formally in the following definition.

11.22 Definition: Let $\phi_k : \mathbf{F}^n \to U_k \subseteq \mathbf{P}^n$ be the k^{th} inclusion map. Given a projective variety $X \subseteq \mathbf{P}^n$, the k^{th} **affine variety in** X is defined to be the set

$$X_k = {\phi_k}^{-1}\big(X \cap U_k\big) \subseteq \mathbf{F}^n .$$

Note that X_k is in fact an affine variety; indeed if $X = V(S) \subseteq \mathbf{P}^n$, where S is a collection of homogeneous polynomials in $\mathbf{F}[x_1, \cdots, x_{n+1}]$, then for $x \in \mathbf{F}^n$ we have $x \in X_k \iff \phi_k(x) \in X \iff f\big(\phi_k(x)\big) = 0$ for all $f \in S \iff g_k(f)(x) = 0$ for all $f \in S$, and so $X_k = V(T) \subseteq \mathbf{F}^n$ where $T = \big\{g_k(f)\big| f \in S\big\}$.

Also, given an affine variety $X \subseteq \mathbf{F}^n$, we define the k^{th} **projective closure of** X to be the projective variety $\overline{\phi_k(X)} \subseteq \mathbf{P}^n$. Usually we fix k and write

$$\overline{X} = \overline{\phi_k(X)} \subseteq \mathbf{P}^n .$$

11.23 Theorem: *Let* $\mathbf{F}$ *be algebraically closed. Let* $f \in \mathbf{F}[x,y]$ *be irreducible and let* $X = V(f) \subseteq \mathbf{F}^2$*. Write* $U = U_3 = \big\{[x,y,z] \in \mathbf{P}^2 \big| z \neq 0\big\}$ *and* $\phi = \phi_3$ *so* $\phi : \mathbf{F}^2 \to U$ *is given by* $\phi(x,y) = [x,y,1]$*, and write* $h = h_3(f) = f\big(\frac{x}{z}, \frac{y}{z}\big)z^{\deg f}$*. Let* $\overline{X} = \overline{\phi(X)} \subseteq \mathbf{P}^2$ *be the projective closure of* X*. Then* $\overline{X} = V(h) \subseteq \mathbf{P}^2$*.*

Proof: Note first that since f is irreducible, its homogenization h must also be irreducible, since if $h = k\ell$ then we would have $f(x,y) = h(x,y,1) = k(x,y,1)\ell(x,y,1)$. Since $\mathbf{F}$ is algebraically closed, the homogeneous variety $V(h) \subseteq \mathbf{F}^{n+1}$ is irreducible, and so the projective variety $V(h) \subseteq \mathbf{P}^2$ is also irreducible.

Now, for $[x,y,1] \in U$ we have $[x,y,1] \in V(h) \subseteq \mathbf{P}^2 \iff h\big([x,y,1]\big) = 0 \iff h(x,y,1) = 0 \iff f(x,y) = 0 \iff (x,y) \in X$. This shows that $V(h) \cap U = \phi(X)$. Since $\phi(X) \subseteq V(h)$ and $V(h)$ is closed, we also have $\overline{\phi(X)} \subseteq V(h)$.

Also, since $\mathbf{P}^2 = U \cup V(z)$, we have $V(h) = \big(V(h) \cap U\big) \cup \big(V(h) \cap V(z)\big) = \phi(X) \cup V(h,z) = \overline{\phi(X)} \cup V(h,z)$. Since $V(h)$ ir irreducible, this implies that $\overline{\phi(X)} = V(h)$ or $V(h,z) = V(h)$. But we cannot have $V(h,z) = V(h)$ since $X = V(f) \neq \emptyset$ (since $\mathbf{F}$ is algebraically closed), so $\phi(X) \neq \emptyset$, so $V(h) \cap U \neq \emptyset$. Thus $V(h) = \overline{\phi(X)}$.