10. Local Dimension and Singularity

10.1 Definition: Let $X \subseteq \mathbf{F}^n$ be a variety and let $a \in X$. Choose a finite set of generators $f_1, \cdots, f_m$ for $I(X)$, and let $f = (f_1, \cdots, f_m)$. We define the (Zarisky) **tangent space of** X **at** a to be

$$T_aX = \ker Df(a) \subseteq \mathbf{F}^n \quad \text{where} Df(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}$$

and we define the (local) **dimension of** X **at** a to be

$$\dim_a X = \dim T_aX \,.$$

Note that $0 \leq \dim_a X \leq n$.

We need to verify that our definition of T_aX does not depend on the choice of generators. Suppose that $I(X) = \langle f_1, \cdots, f_k \rangle = \langle g_1, \cdots, g_\ell \rangle$. For each index j, since $g_j \in \langle f_1, \cdots, f_m \rangle$ we can choose polynomials $p_{j,i} \in \mathbf{F}[x_1, \cdots, x_n]$ such that $g_j = \sum\limits_{i=1}^{m} p_{j,i} f_i$. For all $a \in X$ we have $f_i(a) = 0$ for all i so, by the Product Rule, we have

$$\frac{\partial g_j}{\partial x_k}(a) = \sum_{i=1}^{n} \left(\frac{\partial p_{j,i}}{\partial x_k}(a) f_i(a) + p_{j,i}(a) \frac{\partial f_i}{\partial x_k}(a) \right) = \sum_{i=1}^{n} p_{j,i}(a) \frac{\partial f_i}{\partial x_k}(a)$$

and so $Dg(a) = P(a)Df(a)$ where $P(a)$ is the matrix with entries $p_{j,i}(a)$. It follows that $\ker Df(a) \subseteq \ker Dg(a)$. A similar argument, reversing the roles of f and g, shows that $\ker Dg(a) \subseteq \ker Df(a)$.

10.2 Example: Let $\mathbf{F}$ be an infinite field. Find $\dim_a X$ for each $a \in X$ when X is any one of the varieties $X = V(y^2 - x^3) \in \mathbf{F}^2$, $X = V(y^2 - x^3 - x^2) \subseteq \mathbf{F}^2$, $X = V(z) \cup V(x, y) \subseteq \mathbf{F}^3$.

10.3 Note: For $0 \leq d \leq n$ note that $\dim_a X \leq d \iff \operatorname{rank} Df(a) \geq r$ where $r = n - d$, and we have $\operatorname{rank} Df(a) \geq r$ when at least one of the $r \times r$ submatrices of $Df(a)$ has non-zero determinant. If follows that the set $\left\{ a \in X \middle| \dim_a X > d \right\}$ is the subvariety of X which is cut out by the determinants of the $r \times r$ submatrices of $Df(x)$ (which are all polynomials in x). If we let $m = \min\{\dim_a X | a \in X\}$ and $X_d = \left\{ a \in X \middle| \dim_a X \geq d \right\}$ for $m \leq d \leq n$ then we have a chain of subvarieties $\emptyset \subseteq X_n \subseteq X_{n-1} \subseteq \cdots \subseteq X_{m+1} \subsetneqq X_m = X$. We shall prove below that when $\mathbf{F}$ is algebraically closed and X is irreducible we have

$$\dim(X) = m = \min \left\{ \dim_a X \middle| a \in X \right\}.$$

10.4 Definition: Let $X \subseteq \mathbf{F}^n$ be a variety of pure dimension (meaning that every irreducible component of X has the same dimension). and let $m = \min \left\{ \dim_a X \middle| a \in X \right\}$. For $a \in X$, we say that a is a **non-singular** point of X (or that X is non-singular at a) when $\dim_a X = m$ and we say that a is a **singular** point of X (or that X is singular at a) when $\dim_a X > m$. We denote the set of singular points of X by $\operatorname{Sing}(X)$. Note that $\operatorname{Sing}(X)$ is a proper subvariety of X. We say that X is **non-singular** when $\operatorname{Sing}(X) = \emptyset$.

10.5 Theorem: *Let $X \subseteq \mathbf{F}^n$ and $Y \subseteq \mathbf{F}^m$ be varieties, let $a \in X$, and let $u \in T_aX$.*
(1) If $f : X \to Y$ is a polynomial map then $Df(a)\,u \in T_{f(a)}Y$.
(2) If X and Y are irreducible and $f : X \to Y$ is a rational map which is regular at $a \in X$, then $Df(a)\,u \in T_{f(a)}Y$.

Proof: Suppose that either f is a polynomial map or that X and Y are irreducible and f is a rational map which is regular at a. If f is a polynomial map then let $p = f$, $q = 1$ and $U_q = X$, and if X and Y are irreducible and f is a rational map, then let $f = \frac{p}{q}$ where $p \in \mathbf{F}[x_1, \cdots, x_n]^m$, and $q \in \mathbf{F}[x_1, \cdots, x_n]$ with $q(a) \neq 0$, and let $U_q = \left\{x \in X \middle| q(x) \neq 0\right\}$. Let $I(X) = \langle g_1, \cdots, g_\ell \rangle$ so that $T_aX = \ker Dg(a)$, and let $I(Y) = \langle h_1, \cdots, h_k \rangle$ so that $T_{f(a)}Y = \ker Dh(f(a))$. For all $x \in U_q$ we have $f(x) \in Y$ hence $0 = h_j\big(f(x)\big) = h_j\big(\frac{p(x)}{q(x)}\big)$ for all indices j. Let d be the maximum of the degrees of $h_1, \cdots, h_k$ then, for each index j, let $r_j(x) = q(x)^d h_j\big(\frac{p(x)}{q(x)}\big) \in \mathbf{F}[x_1, \cdots, x_n]$. For all j we have $r_j(x) = 0$ for all $x \in U_q$ and so $r_j \in I(U_q) = I(X) = \langle g_1, \cdots, g_\ell \rangle$. Write $r_j = \sum_{i=1}^{\ell} s_{j,i} g_i$ with each $s_{j,i} \in \mathbf{F}[x_1, \cdots, x_n]$.

For $x \in U_j$ we have $q(x)^d h_j\big(f(x)\big) = r_j(x) = \sum_{i=1}^{\ell} s_{j,i}(x) g_i(x)$. Take the derivative with respect to x_k, using the Product and Chain Rules to get

$$dq(x)^{d-1} \tfrac{\partial q}{\partial x_k}(x) h_j\big(f(x)\big) + q(x)^d \sum_{i=1}^{m} \tfrac{\partial h_j}{\partial y_i}\big(f(x)\big) \tfrac{\partial f}{\partial x_k}(x) = \sum_{i=1}^{\ell} \big(\tfrac{\partial s_{j,i}}{\partial x_k}(x) g_i(x) + s_{j,i}(x) \tfrac{\partial g_i}{\partial x_k}(x)\big).$$

Since $g_i(x) = 0$ and $h_j\big(f(x)\big) = 0$, this simplifies to

$$q(x)^d \sum_{i=1}^{m} \tfrac{\partial h_j}{\partial y_i}\big(f(x)\big) \tfrac{\partial f}{\partial x_k}(x) = \sum_{i=1}^{\ell} s_{j,i}(x) \tfrac{\partial g}{\partial x_k}(x),$$

so we have

$$q(x)^d Dh\big(f(x)\big) Df(x) = s(x) Dg(x)$$

where $s(x)$ is the matrix with entries $s_{j,i}(x)$. In particular, we obtain

$$q(a)^d Dh\big(f(a)\big) Df(a) = s(a) Dg(a).$$

Since $q(a) \neq 0$ and $u \in T_aX = \ker Dg(a)$, we have $Df(a)\,u \in \ker Dh\big(f(a)\big) = T_{f(a)}Y$.

10.6 Definition: Let X and Y be affine varieties and let $a \in X$. If $f : X \to Y$ is a polynomial map, or if X and Y are irreducible and $f : X \to Y$ is a rational map which is regular at a, then the map $f_* : T_aX \to T_{f(a)}Y$ given by $f_*(u) = Df(a)\,u$ is called the **push-forward** of f. The push-forward f_* is sometimes denoted by df or by Df.

10.7 Corollary: *Let X and Y be affine varieties and let $a \in X$.*
(1) If $f : X \to Y$ is a polynomial isomorphism then $f_* : T_aX \to T_{f(a)}Y$ is a vector space isomorphism, and so $\dim_a X = \dim_{f(a)} Y$.
(2) If X and Y are irreducible, and $f : X \to Y$ is a birational equivalence, and f is regualar at a and the inverse of f is regular at $f(a)$, then the pullback $f_* : T_aX \to T_{f(a)}Y$ is a vector space isomorphism, and so $\dim_a X = \dim_{f(a)} Y$.

Proof: This follows from the above theorem together with the observation that when $f : X \to Y$ and $g : Y \to Z$ with f regular at $a \in X$ and g regular at $f(a) \in Y$, we have $(g \circ f)_* = g_* \circ f_*$, along with the observation that the push-forward of the identity map on X is the identity map on T_aX.

10.8 Theorem: *(Generic Dimension of a Hypersurface) Let* $\mathbf{F}$ *be algebraically closed and let* $X \subseteq \mathbf{F}^n$ *be a hypersurface. Then* $\min\left\{ \dim_a X \middle| a \in X \right\} = n - 1$.

Proof: Since $X \in \mathbf{F}^n$ is a hypersurface and $\mathbf{F}$ is algebraically closed, we can choose a non-constant polynomial $f \in \mathbf{F}[x_1, \cdots, x_n]$ such that $X = V(f) \subseteq \mathbf{F}^n$. Say $f = {p_1}^{k_1} {p_2}^{k_2} \cdots {p_\ell}^{k_\ell}$ where the p_i are non-associate irreducible polynomials. Then $I(X) = \sqrt{\langle f \rangle} = \langle g \rangle$ where $g = p_1 p_2 \cdots p_\ell$. For $a \in X$ we have $Dg(a) = \left(\frac{\partial g}{\partial x_1}(a), \cdots, \frac{\partial g}{\partial x_n}(a) \right)$. The rank of $Dg(a)$ is equal to 0 or to 1, and it is equal to 0 if and only if $Dg(a) = 0$, and so $\dim_a X$ is equal to n or to $n-1$, and it is equal to n if and only if $Dg(a) = 0$. It follows that $\min\left\{ \dim_a X \middle| a \in X \right\}$ is equal to n or to $n - 1$, and it is equal to n if and only if $Dg(a) = 0$ for every $a \in X$. Suppose, for a contradiction, that $Dg(a) = 0$ for every $a \in X$. Then for each index k we have $\frac{\partial g}{\partial x_k}(a) = 0$ for all $a \in X$, so that $\frac{\partial g}{\partial x_k} \in I(X) = \langle g \rangle$. Since $\deg_{x_k} \frac{\partial g}{\partial x_k} < \deg_{x_k} g$, and $\frac{\partial g}{\partial x_k} \in \langle g \rangle$ it follows that $\frac{\partial g}{\partial x_k} = 0 \in \mathbf{F}[x_1, \cdots, x_n]$, as a polynomial. When $\operatorname{char}(\mathbf{F}) = 0$, the fact that $\frac{\partial g}{\partial x_k} = 0 \in \mathbf{F}[x_1, \cdots, x_n]$ for all indices k implies that g is constant, but in fact g is not constant. Suppose that $\operatorname{char}(\mathbf{F}) = p$. Then the fact that $\frac{\partial g}{\partial x_k} = 0$ for all k implies that all the exponents of all the variables x_k in g are multiples of p, so g is of the form $g(x) = \sum_{k \in K} c_k {x_1}^{pk_1} {x_2}^{pk_2} \cdots {x_n}^{pk_n}$ where K is a finite set of multi-indices $k = (k_1, \cdots, k_n)$. But then $g = h^p$ where $h(x) = \sum_{k \in K} a_k {x_1}^{k_1} \cdots {x_n}^{k_n}$ with $a_k \in \mathbf{F}$ chosen so that ${a_k}^p = c_k$ (such elements $a_k \in \mathbf{F}$ exist since $\mathbf{F}$ is algebraically closed). This is again not possible, since $g = p_1 p_2 \cdots p_\ell$ where the p_i are non-associate irreducible polynomials.

10.9 Theorem: *(The Separating Transcendence Basis Theorem) Let* $\mathbf{F} \subseteq \mathbf{K}$ *be fields with* $\operatorname{trans}_{\mathbf{F}} \mathbf{K} = r$. *Suppose* $\mathbf{F}$ *is algebraically closed and* $\mathbf{K}$ *is finitely generated over* $\mathbf{F}$. *Let* $S \subseteq \mathbf{K}$ *be a finite set of generators for* $\mathbf{K}$ *over* $\mathbf{F}$. *Then there exist a transcendence basis* $\{u_1, \cdots, u_r\} \subseteq S$ *for* $\mathbf{K}$ *over* $\mathbf{F}$ *such that* $\mathbf{K}$ *is separable over* $\mathbf{F}(u_1, \cdots, u_r)$.

Proof: We sketch a proof. If $\operatorname{char}(\mathbf{F}) = 0$ there is nothing to prove. Suppose that $\operatorname{char}(\mathbf{F}) = p$. Say $S = \{v_1, \cdots, v_n\}$ is a finite set of generators for $\mathbf{K}$ over $\mathbf{F}$. Reorder the elements v_i, if necessary, so that $\{v_1, \cdots, v_r\}$ is a transcendence basis for $\mathbf{K}$ over $\mathbf{F}$. If $\mathbf{K}$ is separable over $\mathbf{F}(v_1, \cdots, v_r)$ then we are done. Suppose that $\mathbf{K}$ is not separable over $\mathbf{F}(v_1, \cdots, v_r)$. Then one of the elements v_i, with $i > r$, is not separable over $\mathbf{F}(v_1, \cdots, v_r)$. Reorder, if necessary, so that v_{r+1} is not separable over $\mathbf{F}(v_1, \cdots, v_r)$. Multiply the minimal polynomial of v_{r+1} over $\mathbf{F}(v_1, \cdots, v_r)$ by the least common denominator to obtain an irreducible polynomial $f \in \mathbf{F}[x_1, \cdots, x_{r+1}]$ with $f(v_1, \cdots, v_r, v_{r+1}) = 0$. Since v_{r+1} is not separable over $\mathbf{F}(v_1, \cdots, v_r)$ it follows that x_{r+1} occurs in f with exponents which are all multiples of p. One of the variables $x_1, \cdots, x_r$ must occur in f with an exponent which is not a multiple of p otherwise, since F is algebraically closed, f would be equal to the p^{th} power of another polynomial, but f is irreducible. Reorder, if necessary, so that x_1 occurs in f with an exponent which is not a multiple of p. Note that since $\{v_2, \cdots, v_{r+1}\}$ is algebraically independent, $f(t, v_2, \cdots, v_{r+1})$ is irreducible in $\mathbf{F}[v_2, \cdots, v_{r+1}][t]$, and v_1 is a root of this irreducible polynomial. Since t occurs in this polynomial with an exponent which is not a multiple of p, it follows that v_1 is separable over $\mathbf{F}(v_2, \cdots, v_{r+1})$. If $\mathbf{K}$ is not separable over $\mathbf{F}(v_2, \cdots, v_{r+1})$ we repeat the above procedure, reordering again if necessary, obtaining a transcendence basis $\{v_3, \cdots, v_{r+2}\}$ for $\mathbf{K}$ over $\mathbf{F}$ such that v_1 and v_2 are both separable over $\mathbf{F}(v_3, \cdots, v_{r+2})$. Eventually, this process will produce a transcendence basis $\{u_1, \cdots, u_r\} = \{v_{1+k}, \cdots, v_{r+k}\}$ with $\mathbf{K}$ separable over $\mathbf{F}(u_1, \cdots, u_r)$, as required.

10.10 Theorem: *(The Primitive Element Theorem) Let* $\mathbf{F} \subseteq \mathbf{K}$ *be fields. Suppose that* $\mathbf{K}$ *is separable and finite dimensional over* $\mathbf{F}$*. Then there exists* $u \in \mathbf{K}$ *such that* $\mathbf{K} = \mathbf{F}[u]$*.*

Proof: If $\mathbf{F}$ is finite then so is $\mathbf{K}$, so the group of units $\mathbf{K}^* = \mathbf{K} \setminus \{0\}$ is a cyclic group, and so $\mathbf{K} = \mathbf{F}[u]$ where u is a generator of $\mathbf{K}^*$. Suppose that $\mathbf{F}$ is infinite. Since $\mathbf{K}$ is finite dimensional over $\mathbf{F}$ it is finitely generated and algebraic. Let $\mathbf{L}$ be the splitting field over $\mathbf{F}$ of the minimal polynomials of a finite set of generators for $\mathbf{K}$ over $\mathbf{F}$. Then we have $\mathbf{F} \subseteq \mathbf{K} \subseteq \mathbf{L}$ and $\mathbf{L}$ is a finite dimensional Galois extension field of $\mathbf{F}$. It follows, from Galois Theory, that $\text{Aut}_{\mathbf{F}}\mathbf{L}$ is a finite group and hence that there exist only finitely many intermediate fields between $\mathbf{F}$ and $\mathbf{L}$. Thus there are only finitely many intermediate fields between $\mathbf{F}$ and $\mathbf{K}$. Choose $u \in \mathbf{K}$ such that the index $\big[\mathbf{F}[u] : \mathbf{F}\big]$ is maximal. We claim that $\mathbf{F}[u] = \mathbf{K}$. Suppose, for a contradiction, that $\mathbf{F}[u] \subsetneqq \mathbf{K}$. Choose $v \in \mathbf{K}$ with $v \notin \mathbf{F}[u]$. Since $\mathbf{F}$ is infinite and there are only finitely many intermediate fields between $\mathbf{F}$ and $\mathbf{K}$ we can choose two distinct elements $a, b \in \mathbf{F}$ such that $\mathbf{F}[u + av] = \mathbf{F}[u + bv]$. Then we have $(a - b)v = (u + av) - (u + bv) \in \mathbf{F}[u + av]$. Since $a - b \neq 0$ this implies that $v \in \mathbf{F}[u + av]$ and hence $u = (u + av) - av \in \mathbf{F}[u + av]$. Since $u \in \mathbf{F}[u + av]$ we have $\mathbf{F}[w] \subseteq \mathbf{F}[u + av] \subseteq \mathbf{K}$. By our choice of u, it follows that $\mathbf{F}[u] = \mathbf{F}[u + av]$. But then we have $v \in \mathbf{F}[u + av] = \mathbf{F}[u]$ giving the desired contradiction.

10.11 Theorem: *Every irreducible variety of dimension* d *is birationally equivalent to a hypersurface in* $\mathbf{F}^{d+1}$*.*

Proof: Let $X \subseteq \mathbf{F}^n$ be an irreducible variety with $\dim(X) = d$. We have $K(X) = \mathbf{F}(x_1, \cdots, x_n)$ where each $x_k \in A(X) \subseteq K(X)$. By the Separating Transcendence Basis Theorem, we can reorder the elements x_k, if necessary, so that $\{x_1, \cdots, x_d\}$ is a transcendence basis for $K(X)$ over $\mathbf{F}$ and $K(X)$ is separable over $\mathbf{F}(x_1, \cdots, x_d)$. By the primitive Element Theorem, we can choose $u \in K(X)$ such that $K(X) = K(x_1, \cdots, x_d)[u]$. Multiply the minimal polynomial of u over $\mathbf{F}(x_1, \cdots, x_d)$ by a common denominator of the coefficients to obtain an irreducible polynomial $p \in \mathbf{F}[x_1, \cdots, x_d, t]$ with $p(x_1, \cdots, x_d, u) = 0$. Let $Y = V(p) \in \mathbf{F}^{d+1}$. Then $A(Y) \cong \mathbf{F}[x_1, \cdots, x_d, t]/\langle f \rangle \cong \mathbf{F}[x_1, \cdots, x_d, u] \subseteq K(X)$, and $K(Y) \cong \mathbf{F}(x_1, \cdots, x_d, u) = K(X)$. Thus $X \sim Y = V(p) \subseteq \mathbf{F}^{d+1}$.

10.12 Theorem: *(Generic Dimension) Let* $\mathbf{F}$ *be algebraically closed and let* $X \subseteq \mathbf{F}^n$ *be a variety of pure dimension. Then* $\dim(X) = \min\big\{\dim_a X \big| a \in X\big\}$*.*

Proof: This follows from Theorem 10.11 together with Corollary 10.7 and Theorem 10.8.