Chapter 4. Topology

Topological Spaces and Bases

4.1 Definition: A **topology** on a set X is a set $\mathcal{T}$ of subsets of X such that
(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
(2) if $\mathcal{S}$ is any subset of $\mathcal{T}$ then $\bigcup \mathcal{S} \in \mathcal{T}$, and
(3) if $\mathcal{S}$ is any finite subset of $\mathcal{T}$ then $\bigcap \mathcal{S} \in \mathcal{T}$.

A set X with a topology $\mathcal{T}$ is called a **topological space**. When X is a topological space with topology $\mathcal{T}$, for a set $A \subseteq X$, we say that A is **open** (in X, with respect to $\mathcal{T}$) when $A \in \mathcal{T}$ and we say that A is **closed** (in X, with respect to $\mathcal{T}$) when $A^c = X \setminus A \in \mathcal{T}$.

When X is a topological space and $A \subseteq X$, the **interior** of A (in X), denoted by A^o or $\mathrm{Int}_X(A)$, is the smallest open set contained in A (that is the union of the set of all open sets in X which are contained in A) and the **closure** of A (in X), denoted by $\overline{A}$ or $\mathrm{Cl}_X(A)$, is the largest closed set which contains A (that is the intersection of the set of all closed sets in X which contain A).

When S and T are two topologies of X, we say that S is **coarser** than T, or that T is **finer** than S, when $S \subseteq T$. We say that S is **strictly coarser** than T, or that T is **strictly finer** than S, when $S \subsetneqq T$.

Recall (or verify) that the intersection of a nonempty set of topologies on X is also a topology on X. Because of this, when X is a set and $\mathcal{S}$ is any set of subsets of X, there is a unique coarsest topology $\mathcal{T}$ on X with $\mathcal{S} \subseteq \mathcal{T}$ (namely the intersection of the set of all topologies on X which contain $\mathcal{S}$) which we call the topology on X **generated by** $\mathcal{S}$. As an exercise, you can verify that this topology is equal to the set of arbitrary unions of finite intersections of elements in S (where an empty intersection is equal to X and an empty union is equal to $\emptyset$).

A **basis** for a topology on a set X is a set $\mathcal{B}$ of subsets of X such that
(1) $X = \bigcup \mathcal{B}$, and
(2) for all $U, V \in \mathcal{B}$ and $a \in U \cap V$, there exists $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$.

When $\mathcal{B}$ is a basis for a topology on X and $\mathcal{T}$ is the topology on X generated by $\mathcal{B}$ (so the elements of $\mathcal{B}$ are open in X with respect to $\mathcal{T}$), we say that $\mathcal{B}$ is a basis for the topology $\mathcal{T}$, and the elements in $\mathcal{B}$ are called **basic open sets** in X (or in $\mathcal{T}$).

4.2 Theorem: *Let X be a set, let $\mathcal{B}$ be a basis for a topology on X, and let T be the topology generated by $\mathcal{B}$ on X. Then for all $A \subseteq X$ we have*

(1) $A \in \mathcal{T}$ if and only if for every $a \in A$ there exists $U \in \mathcal{B}$ such that $a \in U \subseteq A$, and

(2) $A \in \mathcal{T}$ if and only if A is a union of elements of $\mathcal{B}$.

Proof: Let $\mathcal{S} = \left\{ A \subseteq X \,\middle|\, \forall a \in A\ \exists U \in \mathcal{B}\ \ a \in U \subseteq A \right\}$. We claim that $\mathcal{S}$ is a topology on X. Note that $\emptyset \in \mathcal{S}$ (vacuously) and $X \in \mathcal{S}$ (because $X = \bigcup \mathcal{B}$, so given $a \in X$ we can choose $U \in \mathcal{B}$ with $a \in U$). When $\mathcal{R}$ is any subset of $\mathcal{S}$, given $a \in \bigcup R$ we can choose $U \in \mathcal{R}$ with $a \in U$ and then we have $a \in U \in \mathcal{R}$ showing that $\bigcup R \in \mathcal{S}$. It remains to show that $\bigcap R \in \mathcal{S}$ for every finite set $\mathcal{R} \subseteq \mathcal{S}$. By induction, it suffices to show that for all $A, B \in \mathcal{S}$ we have $A \cap B \in \mathcal{S}$. Let $A, B \in \mathcal{S}$. Let $a \in A \cap B$. Since $a \in A$ and $A \in \mathcal{S}$ we can choose $U \in \mathcal{B}$ with $a \in U \subseteq A$. Since $a \in V$ and $V \in \mathcal{S}$, we can choose $V \in \mathcal{B}$ such that $b \in V \subseteq B$. Since $\mathcal{B}$ is a basis, we can choose $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$. Then we have $a \in W \subseteq U \cap V \subseteq A \cap B$, and so $A \cap B \in \mathcal{S}$. Thus $\mathcal{S}$ is a topology on X, as claimed.

We claim that for $A \subseteq X$, we have $A \in \mathcal{S}$ if and only if A is a union of elements in $\mathcal{B}$. If $A \in \mathcal{S}$ then for each $a \in A$ we can choose $U_a \in \mathcal{B}$ such that $a \in U_a \subseteq A$, and then we have $A = \bigcup_{a \in A} U_a$, which is a union of elements of $\mathcal{B}$. If, on the other hand, A is a union of elements of $\mathcal{B}$, say $A = \bigcup \mathcal{R}$ where $\mathcal{R} \subseteq \mathcal{B}$, then given $a \in A$ we can choose $U \in \mathcal{R}$ such that $a \in U$, and then we have $a \in U \subseteq A$, showing that $A \in \mathcal{S}$.

Finally, we claim that $\mathcal{S} = \mathcal{T}$. Note that when $U \in \mathcal{B}$ we have $U \in \mathcal{S}$ (for the deep reason that when $a \in U$ we have $a \in U \subseteq U$). Since $\mathcal{S}$ is a topology which contains $\mathcal{B}$ and $\mathcal{T}$ is the coarsest topology which contains $\mathcal{B}$, we have $\mathcal{T} \subseteq \mathcal{S}$. Since every topology which contains $\mathcal{B}$ also contains all possible unions of elements in $\mathcal{B}$, it follows that $\mathcal{T}$ contains all such unions, and so $\mathcal{S} \subseteq \mathcal{T}$. Thus we have $\mathcal{S} = \mathcal{T}$, as claimed, and we have proven both parts of the theorem.

4.3 Example: In a metric space X, the set $\mathcal{B} = \big\{ B(a,r) \,\big|\, a \in X, r > 0 \big\}$ is a basis for the metric topology on X.

4.4 Theorem: *Let X be a topological space with basis $\mathcal{B}$, and let $A \subseteq X$. Then for $a \in X$ we have $a \in \overline{A}$ if and only if $A \cap U \neq \emptyset$ for every $U \in \mathcal{B}$ with $a \in U$.*

Proof: For $K \subseteq X$, K is closed with $A \subseteq K$ if and only if K^c is open with $A \cap K^c = \emptyset$. Since $\overline{A} = \bigcap \big\{ K \subseteq X \,\big|\, K \text{ is closed}, A \subseteq K \big\}$, we have $\overline{A}^{\,c} = \bigcup \big\{ V \subseteq X \,\big|\, V \text{ is open}, A \cap V = \emptyset \big\}$. Thus $a \notin \overline{A}$ if and only if there exists an open set $V \subseteq X$ with $A \cap V = \emptyset$ such that $a \in V$. Equivalently, $a \in \overline{A}$ if and only if for every open set $V \subseteq X$ with $a \in V$ we have $A \cap V \neq \emptyset$.

When $a \in \overline{A}$ so that $A \cap V \neq \emptyset$ for every open set $V \subseteq X$ with $a \in V$, it is immediate that $A \cap U \neq \emptyset$ for every $U \in \mathcal{B}$ with $a \in U$. Suppose that $A \cap U \neq \emptyset$ for every $U \in \mathcal{B}$ with $a \in U$. Given an open set $V \subseteq X$ with $a \in V$, we can choose a basic open set $U \in \mathcal{B}$ with $a \in U \subseteq V$ and then we have $A \cap U \neq \emptyset$ hence also $A \cap V \neq \emptyset$. Thus $a \in \overline{A}$, as required.

4.5 Example: When Y is a topological space with topology $\mathcal{T}$ and $X \subseteq Y$, the **subspace topology** on X is the topology $\mathcal{S} = \big\{ V \cap X \,\big|\, V \in \mathcal{T} \big\}$. Verify that a subset $A \subseteq X$ is closed in X if and only if there exists a closed set B in Y such that $A = B \cap X$. Verify that if $\mathcal{C}$ is a basis for the topology $\mathcal{T}$ on Y, then $\mathcal{B} = \big\{ V \cap X \,\big|\, V \in \mathcal{B} \big\}$ is a basis for the subspace topology $\mathcal{S}$ on X. Also, recall (or verify as an exercise) that in the case that Y is a metric space and $\mathcal{T}$ is the metric topology on Y, the subspace topology on X is equal to the topology on X induced by the metric on X obtained by restricting the metric on Y.

4.6 Example: When X and Y are topological spaces with topologies $\mathcal{S}$ and $\mathcal{T}$, the **product topology** on $X \times Y$ is the topology with basis $\mathcal{E} = \big\{ U \times V \,\big|\, U \in \mathcal{S}, V \in \mathcal{T} \big\}$. Verify that $\mathcal{E}$ is in fact a basis for a topology on $X \times Y$, and verify that when $\mathcal{B}$ and $\mathcal{C}$ are bases for the topologies on X and Y, the set $\mathcal{D} = \big\{ U \times V \,\big|\, U \in \mathcal{B}, V \in \mathcal{C} \big\}$ is another basis for the product topology on $X \times Y$. Also verify, as an exercise, that when $A \subseteq X$ and $B \subseteq Y$, the subspace topology on $A \times B$, as a subspace of $X \times Y$ using the product topology, is equal to the product topology on $A \times B$ where A and B use the subspace topologies, as subspaces of X and Y.

4.7 Example: When X is a set and $\sim$ is an equivalence relation on X, recall that the **quotient** of X by $\sim$ is the set of equivalence classes

$$X/\!\sim\, = \big\{ [a] \,\big|\, a \in X \big\} \quad \text{where} \quad [a] = \big\{ x \in X \,\big|\, x \sim a \big\}$$

and the **quotient map** $q : X \to X/\!\sim$ is the map given by $q(a) = [a]$. When X is a topological space with topology $\mathcal{T}$, the **quotient topology** on $X/\!\sim$ is the topology

$$\mathcal{S} = \Big\{ V \subseteq X/\!\sim \,\Big|\, q^{-1}(V) \in \mathcal{T} \Big\} = \Big\{ V \subseteq X/\!\sim \,\Big|\, \bigcup V \in \mathcal{T} \Big\}.$$

Continuous Functions and Compact Sets

4.8 Definition: A topological space X is called **Hausdorff** when it has the property that for all $a, b \in X$ with $a \neq b$ there exist disjoint open sets $U, V \subseteq X$ wigth $a \in U$ and $b \in V$. Note that when X is Hausdorff and $a \in X$, the set $\{a\}$ is closed.

4.9 Example: All metric spaces are Hausdorff because given $a \neq b$ we can let $r = d(a, b)$ and take $U = B\big(a, \frac{r}{2}\big)$ and $V = B\big(b, \frac{r}{2}\big)$.

4.10 Definition: Let X and Y be topological spaces. A function $f : X \to Y$ is called **continuous** when $f^{-1}(V)$ is open in X for every open set $V \subseteq Y$. Equivalently, f is continuous if and only if $f^{-1}(B)$ is closed in X for every closed set $B \subseteq Y$.

4.11 Definition: Let X be a topological space and let $A \subseteq X$. An **open cover** for A (in X) is a set S of open sets in X such that $A \subseteq \bigcup S$. When S is an open cover for A in X, a **subcover** of S for A is a subset $T \subseteq S$ such that $A \subseteq \bigcup T$. We say that A is **compact** (in X) when every open cover for A has a finite subcover.

4.12 Theorem: *Let $A \subseteq X \subseteq Y$ where Y is a topological space. Then A is compact in X (where X uses the subspace topology inherited from Y) if and only if A is compact in Y.*

Proof: Suppose that A is compact in X. Let T be an open cover for A in Y. For each $V \in T$, let $U_V = V \cap X$ and note that U_V is open in X, using the subspace topology. Since $A \subseteq X$ and $A \subseteq \bigcup_{V \in T} V$, we also have $A \subseteq \bigcup_{V \in T}(V \cap X) = \bigcup_{V \in T} U_V$. Thus $S = \big\{U_V \big| V \in T\big\}$ is an open cover for A in X. Since A is compact in X we can choose a finite subcover, say $\big\{U_{V_1}, \cdots U_{V_n}\big\}$ of S, where each $V_i \in T$. Since $A \subseteq \bigcup_{i=1}^n U_{V_i} = \bigcup_{i=1}^n (V_i \cap X)$, we also have $A \subseteq \bigcup_{i=1}^n V_i$ and so $\{V_1, \cdots, V_n\}$ is a finite subcover of T.

Suppose, conversely, that A is compact in Y. Let S be an open cover for A in X. For each $U \in S$, since X is using the subspace topology we can choose an open set V_U in Y such that $U = V_U \cap X$. Then $T = \big\{V_U \big| U \in S\big\}$ is an open cover of A in Y. Since A is compact in Y we can choose a finite subcover, say $\big\{V_{U_1}, \cdots, V_{U_n}\big\}$ of T, where each $U_i \in S$. Then we have $A \subseteq \bigcup_{i=1}^n (V_{U_i} \cap X) = \bigcup_{i=1}^n U_i$ and so $\{U_1, \cdots, U_n\}$ is a finite subcover of S.

4.13 Remark: Let $A \subseteq X$ where X is a topological space. By the above theorem, note that A is compact in X if and only if A is compact in itself. For this reason, we do not usually say that A is compact in X, we simply say that A is compact.

4.14 Definition: Let X be a topological space. We say that X has the **finite intersection property on closed sets** when for every set T of closed sets in X, if every finite subset of T has non-empty intersection, then T has non-empty intersection.

4.15 Theorem: *Let X be a topological space. Then X is compact if and only if X has the finite intersection property on closed sets.*

Proof: Suppose that X is compact. Let T be a set of closed sets in X. Suppose that T has empty intersection, that is suppose $\bigcap_{A \in T} A = \emptyset$. Then $\bigcup_{A \in T} A^c = X$ so the set $S = \big\{A^c \big| A \in T\big\}$ is an open cover for X. Since X is compact, we can choose a finite subcover, say $\big\{{A_1}^c, \cdots, {A_n}^c\big\}$ of S for X. Then we have $A_1 \cap A_2 \cap \cdots \cap A_n = \emptyset$, showing that some finite subset of T has empty intersection.

Suppose, conversely, that X has the finite intersection property on closed sets. Let S be an open cover of X. Let $T = \big\{U^c \,\big|\, U \in S\big\}$. Since $\bigcup S = X$ we have $\bigcap T = \big(\bigcup S\big)^c = \emptyset$. Since X has the finite intersection on closed sets, there exists a finite subset of T with empty intersection. so we can choose $U_1, U_2, \cdots U_n \in S$ such that ${U_1}^c \cap \cdots \cap {U_n}^c = \emptyset$. It follows that $U_1 \cup \cdots \cup U_n = X$, so S has a finite subcover.

4.16 Theorem: *Every closed subspace of a compact space is compact.*

Proof: Suppose that X is compact and $A \subseteq X$ is closed. Let S be an open cover for A. Then $S \cup \{A^c\}$ is an open cover for X. Since X is compact, we can choose a finite subcover T of $S \cup \{A^c\}$. Note that T may or may not contain the set A^c but, in either case, $T \setminus \{A^c\}$ is an open cover for A with $T \setminus \{A^c\} \subseteq S$, so that $T \setminus \{A^c\}$ is a finite subcover of S.

4.17 Theorem: *Every compact subspace of a Hausdorff space is closed.*

Proof: Suppose X is Hausdorff and $A \subseteq X$ is compact. Let $b \in A^c = X \setminus A$. For each $a \in A$, since X is Hausdorff we can choose disjoint open sets $U_a, V_a \subseteq X$ with $a \in U_a$ and $b \in V_a$. Since $\mathcal{S} = \{U_a \,\big|\, a \in A\}$ is an open cover of A, and A is compact, we can choose a finite subcover of X, so we can choose $a_1, a_2, \cdots, a_n \in A$ such that $A \subseteq U_{a_1} \cup \cdots \cup U_{a_n}$. The sets $U = U_{a_1} \cup \cdots \cup U_{a_n}$ and $V = V_{a_1} \cap \cdots \cap V_{a_n}$ are disjoint open sets with $A \subseteq U$ and $b \in V$. This shows that for every $b \in A^c$ there is an open set $V = V_b$ with $b \in V_b \subseteq A^c$. Thus A^c is open (it is the union of the open sets V_b) and hence A is closed.

4.18 Theorem: *The image of a compact space under a continuous map is compact.*

Proof: Suppose that X is compact and $f : X \to Y$ is continuous. Let T be an open cover for $f(X)$ in Y. Since f is continuous, so that $f^{-1}(V)$ is open in X for each $V \in T$, the set $S = \{f^{-1}(V) \big| V \in T\}$ is an open cover for X. Since X is compact, we can choose a finite subcover, say $\{f^{-1}(V_1), f^{-1}(V_2), \cdots, f^{-1}(V_n)\}$ of S, with each $V_i \in T$. Then the set $\{V_1, V_2, \cdots, V_n\}$ is a finite subcover of T for $f(X)$.

4.19 Example: Note that continuous maps do not necessarily send closed sets to closed sets. For example, the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{2}{\pi} \tan^{-1}(x)$ sends the closed set $\mathbb{R}$ homeomorphically to the open interval $(-1, 1)$.

4.20 Theorem: *(The Extreme Value Theorem) A continuous map $f : X \to \mathbb{R}$ defined on a compact space X attains its maximum and minimum values.*

Proof: Suppose X is compact and $f : X \to \mathbb{R}$ is continuous. Since $f(X)$ is compact, it is closed and bounded in $\mathbb{R}$. Since $f(X)$ is bounded in $\mathbb{R}$, it follows that $m = \inf f(X)$ and $M = \sup f(X)$ are both finite real numbers, and since $f(X)$ is closed in $\mathbb{R}$ it follows that $m \in f(X)$ and $M \in f(X)$ so that we can choose $a, b \in X$ such that $f(a) = m = \inf f(X)$ and $f(b) = M = \sup f(X)$.

4.21 Theorem: *Let X and Y be topological spaces with X compact and Y Hausdorff. Let $f : X \to Y$ be continuous and bijective. Then f is a homeomorphism.*

Proof: Let $g = f^{-1} : Y \to X$. We need to prove that g is continuous. Let $A \subseteq X$ be closed in X. Since X is compact and $A \subseteq X$ is closed, it follows (from Theorem 4.12) that A is compact. Since the map $f : A \to Y$ is continuous and A is compact, it follows (from Theorem 4.14) that $f(A)$ is compact. Since $f(A)$ is compact and Y is Hausdorff, it follows (from Theorem 4.13) that $f(A)$ is closed. Since $g = f^{-1}$ we have $g^{-1}(A) = f(A)$, which is closed. Since $g^{-1}(A)$ is closed in Y for every closed set A in X, it follows (by taking complements) that $g^{-1}(U)$ is open in Y for every open set U in X. Thus g is continuous.

4.22 Example: In the above theorem, the requirement that X is compact is necessary. For example, if X is the interval $X = [0, 2\pi)$ and Y is the unit circle $Y = \{z \in \mathbb{C} \big| \|z\| = 1\}$, then the map $f : X \to Y$ given by $f(t) = e^{i\,t}$ is continuous and bijective, but the inverse map is not continuous at 1.

Urysohn's Lemma and The Tietze Extension Theorem

4.23 Definition: A topological space X is called **normal** when all one-point sets are closed in X, and for all disjoint closed sets $A, B \subseteq X$ there exist disjoint open sets $U, V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$.

4.24 Example: Recall (or verify) that all metric spaces are normal.

4.25 Theorem: *(Urysohn's Lemma) Let X be a normal topological space. For any disjoint closed sets $A, B \subseteq X$ there exists a continuous map $f : X \to [0,1]$ with $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.*

Proof: Let $A, B \subseteq X$ be closed. Say $[0,1] \cap \mathbb{Q} = \{a_0, a_1, a_2, a_3, \cdots\}$ where the terms a_k are distinct with $a_0 = 0$ and $a_1 = 1$. Choose disjoint open sets $U_0, V_0 \subseteq X$ with $A \subseteq U_0$ and $B \subseteq V_0$. Note that

$$U_0 \cap V_0 = \emptyset \Longrightarrow U_0 \subseteq {V_0}^c \Longrightarrow \overline{U}_0 \subseteq {V_0}^c \Longrightarrow \overline{U}_0 \subseteq B^c.$$

Let $U_1 = B^c$ so that $A \subseteq U_0 \subseteq \overline{U}_0 \subseteq U_1 = B^c$. Let $n \geq 2$ and suppose, inductively, that we have defined open sets $U_{a_0}, U_{a_1}, \cdots U_{a_{n-1}}$ such that when $a_k < a_\ell$ we have $\overline{U}_{a_k} \subseteq U_{a_\ell}$. Define U_{a_n} as follows. Rearrange the terms in the set $\{a_0, a_1, \cdots, a_n\}$ in increasing order and say $a_k < a_n < a_\ell$ are consecutive. Since $\overline{U}_{a_k} \subseteq U_{a_\ell}$, we have $\overline{U}_{a_k} \cap {U_{a_\ell}}^c = \emptyset$, so we can choose disjoint open sets $U_{a_n}, V_{a_n} \subseteq X$ with $\overline{U}_{a_k} \subseteq U_{a_n}$ and $U_{a_\ell}^c \subseteq V_{a_n}$, and then

$$U_{a_n} \cap V_{a_n} = \emptyset \Longrightarrow U_{a_n} \subseteq {V_{a_n}}^c \Longrightarrow \overline{U}_{a_n} \subseteq V_{a_n}^c \subseteq U_{a_\ell}.$$

Recursively, we have defined U_{a_n} for all $n \geq 0$, so we have defined U_r for all $r \in [0,1] \cap \mathbb{Q}$. For $r \in \mathbb{Q}$ with $r < 0$ we define $U_r = \emptyset$, and for $r \in \mathbb{Q}$ with $r > 1$ we define $U_r = X$, and then we have defined U_r for all $r \in \mathbb{Q}$ so that whenever $r < s$ we have $\overline{U}_r \subseteq U_s$.

Define $f : X \to [0,1]$ by

$$f(x) = \inf \left\{ r \in \mathbb{Q} \,\middle|\, x \in U_r \right\}$$

Note that f does take values in $[0,1]$: indeed for all $x \in X$, we have $f(x) \geq 0$ because $r < 0 \Longrightarrow U_r = \emptyset \Longrightarrow x \notin U_r$, and we have $f(x) \leq 1$ because $r > 1 \Longrightarrow U_r = X \Longrightarrow x \in U_r$. Also note that when $x \in A$ we have $x \in U_0$ so that $f(x) = 0$ and when $x \in B$ and $r \leq 1$ we have $U_r \subseteq U_1 = B^c$ so that $x \notin U_r$, and so $f(x) = 1$.

It remains to show that f is continuous. We shall show that the inverse image of every open interval is open. Let $c, d \in \mathbb{R}$ with $c < d$. Let $a \in f^{-1}(c,d)$ so we have $c < f(a) < d$. Choose $r, s \in \mathbb{Q}$ with $c < r < f(a) < s < d$. We claim that $a \in U_s \setminus \overline{U}_r \subseteq f^{-1}(c,d)$. First we make two observations: for $x \in X$ and $p \in \mathbb{Q}$,

(1) if $x \in \overline{U}_p$ then $x \in U_r$ for all $r > p$ and so $f(x) \leq p$, and
(2) if $x \notin U_p$ then $x \notin U_r$ for any $r \leq p$ and so $f(x) \geq p$.

Since $r < f(a)$ it follows from the first observation that $a \notin \overline{U}_r$, and since $f(a) < s$ it follows from the second observation that $a \in U_s$, and this shows that $a \in U_s \setminus \overline{U}_r$. On the other hand, when $x \in U_s \setminus \overline{U}_r$, since $x \in U_s$ it follows from the first observation that $f(x) \leq s$, and since $x \notin \overline{U}_r$ it follows from the second observation that $f(x) \geq r$, and so we have $f(x) \in [r,s] \subseteq (c,d)$. Thus we have $a \in U_s \setminus \overline{U}_r \subseteq f^{-1}(c,d)$, as claimed. Since $U_s \setminus \overline{U}_r$ is open, we can choose a basic open set V with $a \in V \subseteq U_s \setminus \overline{U}_r \subseteq f^{-1}(c,d)$. Since for every $a \in f^{-1}(c,d)$ there is a basic open set V with $a \in V \subseteq f^{-1}(c,d)$, it follows that $f^{-1}(c,d)$ is open, so that f is continuous, as required.

4.26 Theorem: *(The Tietze Extension Theorem) Let X be a normal topological space, let $A \subseteq X$ be closed, and let $a, b \in \mathbb{R}$ with $a < b$.*

(1) Every continuous map $f : A \to [a, b]$ can be extended to a continuous map $g : X \to [a, b]$.
(2) Every continuous map $f : A \to (a, b)$ can be extended to a continuous map $g : X \to (a, b)$.

Proof: Note that since $[a, b]$ is homeomorphic to the interval $[-1, 1]$, we may replace $[a, b]$ by $[-1, 1]$. Suppose that $f : A \to [-1, 1]$ is continuous.

We begin with an observation. If $h : A \to [-r, r]$ is continuous, then $h^{-1}\big([-r, -\frac{r}{3}]\big)$ and $h^{-1}\big([\frac{r}{3}, r]\big)$ are disjoint closed sets in X, so by scaling and translating the map given by Urysohn's Lemma, we can construct a map $g : X \to \big[-\frac{r}{3}, \frac{r}{3}\big]$ with $g(x) = -\frac{r}{3}$ for all $x \in h^{-1}\big([-r, -\frac{r}{3}]\big)$ and $g(x) = \frac{r}{3}$ for all $x \in h^{-1}\big([\frac{r}{3}, r]\big)$. We then have $\big|g(x)\big| \leq \frac{r}{3}$ for all $x \in X$, and we have $\big|h(x) - g(x)\big| \leq \frac{2r}{3}$ for all $x \in A$.

Since $f : A \to [-1, 1]$ is continuous, by the above observation we can construct a continuous map $g_1 : X \to \big[-\frac{1}{3}, \frac{1}{3}\big]$ such that $\big|f(x) - g_1(x)\big| \leq \frac{2}{3}$ for all $x \in A$. Since $(f - g_1) : A \to \big[-\frac{2}{3}, \frac{2}{3}\big]$ is continuous, we can apply the obove observation again to construct a continuous map $g_2 : X \to \big[-\frac{2}{9}, \frac{2}{9}\big]$ such that $\big|f(x) - g_1(x) - g_2(x)\big| \leq \frac{4}{9}$ for all $x \in A$. Repeating this procedure, we construct maps $g_k : X \to \big[-\frac{2^{k-1}}{3^k}, \frac{2^{k-1}}{3^k}\big]$ such that $\big|f(x) - \sum_{k=1}^{n} g_k(x)\big| \leq \frac{2^n}{3^n}$ for all $x \in A$. Since $\big|g_k(x)\big| \leq \frac{2^{k-1}}{3^k}$ for all $x \in X$, the series $\sum_{k=1}^{\infty} g_k$ converges uniformly on X by the Weierstrass M-Test. Define $g(x) = \sum_{k=1}^{\infty} g_k(x)$ for all $x \in X$. Note that g is continuous by uniform convergence, note that for all $x \in X$ we have $\big|g(x)\big| \leq \sum_{k=1}^{\infty} |g_k(x)| \leq \sum_{k=1}^{\infty} \frac{2^{n-1}}{3^n} = 1$ so that $g : X \to [-1, 1]$, and note that for all $x \in A$, since $\big|f(x) - \sum_{k=1}^{n} g_k(x)\big| \leq \frac{2^n}{3^n}$ we have $f(x) = \sum_{k=1}^{\infty} g_k(x) = g(x)$, and so g extends f. This completes the proof of Part 1.

To prove Part 2, suppose that $f : A \to (a, b)$ is continuous. Note that f is also continuous as a map $f : A \to [a, b]$ so, by Part 1, we can extend f to a continuous map $h : X \to [a, b]$. Let $B = h^{-1}(a) \cup h^{-1}(b)$ and note that B is closed in X and B is disjoint from A. By Urysohn's Lemma, we can construct a continuous map $k : X \to [0, 1]$ with $k(x) = 0$ for all $x \in B$ and $k(x) = 1$ for all $x \in A$. Then $g = kh : X \to (a, b)$ is continuous on X with $g(x) = h(x) = f(x)$ for all $x \in A$.

Infinite Products and Tychanoff's Theorem

4.27 Definition: Let $(X_k)_{k\in K}$ be an indexed set of sets. The **cartesian product** of this indexed set is the set

$$\begin{aligned}\prod_{k\in K} X_k &= \left\{a : K \to \bigcup_{k\in K} X_k \,\middle|\, a(k)\in X_k \text{ for all } k\in K\right\}\\ &= \left\{(a_k)_{k\in K} \,\middle|\, a_k\in X_k \text{ for all } k\in K\right\}.\end{aligned}$$

For each $\ell \in K$ we have the **projection map** $p_\ell : \prod\limits_{k\in K} X_k \to X_\ell$ given by $p_\ell\big((a_k)_{k\in K}\big) = a_\ell$.

When $K = \{1, 2, \cdots, n\}$ we write

$$(a_k)_{k\in K} = (a_1, a_2, \cdots, a_n) \text{ and } \prod_{k\in K} X_k = \prod_{k=1}^{n} X_k = X_1 \times X_2 \times \cdots \times X_n.$$

When $K = \mathbf{Z}^+$ we write

$$(a_k)_{k\in K} = (a_k)_{k\geq 1} = (a_1, a_2, a_3, \cdots) \text{ and } \prod_{k\in K} X_k = \prod_{k=1}^{\infty} X_k = X_1 \times X_2 \times X_3 \times \cdots.$$

When each X_k is a topological space with topology $\mathcal{T}_k$, the **box topology** on the cartesian product is the topology with basis

$$\mathcal{B} = \left\{ \prod_{k\in K} U_k \,\middle|\, U_k \in \mathcal{T}_k \right\}$$

and the **product topology** on the cartesian product is the topology with basis

$$\mathcal{P} = \left\{ \prod_{k\in K} U_k \,\middle|\, U_k \in \mathcal{T}_k \text{ with } U_k = X_k \text{ for all but finitely many } k \in K \right\}.$$

Unless otherwise stated, we shall always assume that $\prod\limits_{k\in K} X_k$ is given the product topology. Note that when the index set K is finite, the box and product topologies are the same, and when K is infinite, the box topology is finer than the product topology.

4.28 Theorem: *Let X_k be a topological space and let $A \subseteq X_k$ be a subspace for each $k \in K$, and let $\prod\limits_{k\in K} X_k$ be given the product topology.*

(1) If each X_k is Hausdorff then so is $\prod\limits_{k\in K} X_k$.

(2) On $\prod\limits_{k\in K} A_k \subseteq \prod\limits_{k\in K} X_k$, the product topology is equal to the subspace topology.

(3) We have $\overline{\prod\limits_{k\in K} A_k} = \prod\limits_{k\in K} \overline{A_k}$.

Analogous results hold when $\prod\limits_{k\in K} X_k$ and $\prod\limits_{k\in K} A_k$ are given the box topology.

Proof: The proof is left as an exercise.

4.29 Theorem: *Let X_k be a topological space for each $k \in K$, and let $\prod\limits_{k\in K} X_k$ be given the product topology. For every topological space A and for every function $f : A \to \prod\limits_{k\in K} X_k$, f is continuous if and only if $f_\ell : A \to X_k$ given by $f_\ell(x) = f(x)_\ell$ is continuous for all $\ell \in K$.*

Proof: For each $\ell \in K$, the projection map $p_\ell : \prod_{k\in K} X_k \to X_\ell$ is continuous because when $U \subseteq X_\ell$ is open, ${p_\ell}^{-1}(U) = \left\{(x_k)_{k\in K} \,\middle|\, x_\ell \in U_\ell\right\}$, which is a basic open set in $\prod_{k\in K} X_k$. Thus if f is continuous then so is each component map $f_\ell : A \to X_\ell$ because $f_\ell = p_\ell \circ f$.

Suppose that each component map $f_\ell : A \to X_\ell$ is continuous. Note that since every open set in $\prod_{k\in K} X_k$ is a union of basic open sets, in order to prove that f is continuous it suffices to prove that the inverse image of every basic open set is open. Let V be any basic open set in $\prod\limits_{k\in K} X_k$, say $V = \prod\limits_{k\in K} U_k$ where each $U_k \subseteq X_k$ is open with $U_k = X_k$ for all but finitely many $k \in K$. Say $U_k = X_k$ for all $k \notin F$ where F is a finite subset of K. Then we have

$$f^{-1}(V) = \left\{a\in A \,\middle|\, f(a)_\ell \in U_\ell \text{ for all } \ell\in F\right\} = \bigcap_{\ell\in F} {f_\ell}^{-1}(U_\ell)$$

which is open in A.

4.30 Example: When $\mathbb{R}^\omega = \prod_{k=1}^\infty \mathbb{R}$ is given the box topology, and $f : \mathbb{R} \to \mathbb{R}^\omega$ is given by $f(t) = (t, t, t, \cdots)$, the component maps, given by $f_\ell(t) = t$, are all continuous, but the function f is not: indeed for the basic open set $V = \prod_{k=1}^\infty \left(-\frac{1}{k}, \frac{1}{k}\right)$, we have $f^{-1}(V) = \{0\}$.

4.31 Theorem: *(Tychanoff's Theorem) The product of any indexed set of compact spaces is compact, using the product topology.*

Proof: Let X_k be compact for each $k \in K$. We shall prove that $\prod X_k$ has the finite intersection property on closed sets. Let T be a set of closed sets in $\prod X_k$ such that every finite subset of T has non-empty intersection. We need to show that $\bigcap T \neq \emptyset$. By Zorn's Lemma, we can choose a maximal set S of subsets of $\prod X_k$ with $T \subseteq S$ such that every finite subset of S has non-empty intersection $\big($let $\mathcal{R}$ be the set of all such sets S and note that for every chain $\mathcal{C}$ in $\mathcal{R}$ we have $\bigcup \mathcal{C} \in \mathcal{R}\big)$. Note that the maximality of S implies that S is closed under finite intersection (since if $A_1, \cdots, A_n \in S$ then every intersection of a finite subset of $S \cup \{A_1 \cap \cdots \cap A_n\}$ is also an intersection of a finite subset of $S\big)$.

We shall show that $\bigcap \left\{\overline{A} \,\middle|\, A\in S\right\} \neq \emptyset$, hence $\bigcap T \neq \emptyset$ since if $A \in T$ then $A = \overline{A} \in S$. Let $k \in K$. Note that finite subsets of $\left\{p_k(A) \,\middle|\, A \in S\right\}$ have non-empty intersection $\big($because if $A_1, \cdots, A_n \in S$ then $p_k(A_1) \cap \cdots \cap p_k(A_n) = p_k(A_1 \cap \cdots \cap A_n) \neq \emptyset\big)$, and hence finite subsets of $\left\{\overline{p_k(A)} \,\middle|\, A\in S\right\}$ also have nonempty intersection. Since X_k is compact, so X_k has the finite intersection property on closed sets, it follows that $\bigcap \left\{\overline{p_k(A)} \middle| A\in S\right\} \neq \emptyset$, so we can choose $a_k \in X_k$ such that $a_k \in \overline{p_k(A)}$ for every $A \in S$. We do this for each $k \in K$, that is for each $k \in K$ we choose $a_k \in X_k$ with $a_k \in \overline{p_k(A)}$ for every $A \in S$, then we let $a = (a_k)_{k\in K} \in \prod\limits_{k\in K} X_k$.

We claim that $a \in \overline{A}$ for every $A \in S$. Let $k \in K$. Let U_k be an open set in X_k with $a_k \in U_k$. Then for every $A \in S$, we have $a_k \in \overline{p_k(A)} \cap U_k$ so that $\overline{p_k(A)} \cap U_k \neq \emptyset$ hence $p_k(A) \cap U_k \neq \emptyset$ $\big($if we had $p_k(A) \cap U_k = \emptyset$ then $p_k(A) \subseteq {U_k}^c$ hence $\overline{p_k(A)} \subseteq {U_k}^c$ so that $\overline{p_k(A)} \cap U_k = \emptyset\big)$. For each $A \in S$, since $p_k(A) \cap U_k \neq \emptyset$, we can choose $b \in A$ such that $p_k(b) \in U_k$, that is $b \in {p_k}^{-1}(U_k)$, and hence ${p_k}^{-1}(U_k) \cap A \neq \emptyset$. Since S is closed under finite intersection and ${p_k}^{-1}(U_k) \cap A \neq \emptyset$ for every $A \in S$, the maximality of S implies that ${p_k}^{-1}(U_k) \in S$. Let V be any basic open set in $\prod X_k$ with $a \in V$, say $V = \prod U_k$ where each $U_k \subseteq X_k$ is open with $a_k \in U_k$, and with $U_k = X_k$ for all $k \in F$ where F is a finite subset of K. Since ${p_k}^{-1}(U_k) \in S$ for every $k \in K$ and S is closed under finite intersection, we have

$$V = \left\{(x_k)_{k\in K} \,\middle|\, x_k \in U_k \text{ for all } k \in F\right\} = \bigcap_{k\in F} {p_k}^{-1}(U_k) \in S.$$

Since $V \in S$ and every finite subset of S has non-empty intersection, we have $A \cap V \neq \emptyset$ for all $A \in S$. Given $A \in S$, since $A \cap V \neq \emptyset$ for every basic open set V in $\prod X_k$ with $a \in V$, it follows that $a \in \overline{A}$. Thus $a \in \overline{A}$ for all $A \in S$, so $\bigcap \left\{\overline{A} \,\middle|\, A\in S\right\} \neq \emptyset$, as required.

Nets

4.32 Definition: A **directed set** is a set K together with a binary relation $\leq$ such that
(1) for all $a \in X$ we have $a \leq a$,
(2) for all $a, b, c \in X$, if $a \leq b$ and $b \leq c$ then $a \leq c$, and
(3) for all $a, b \in X$ there exists $c \in X$ such that $a \leq c$ and $b \leq c$.

When $a \leq b$ we also write $b \geq a$. A **net** in a topological space X is an indexed set $(x_k)_{k\in K}$ in X whose index set K is a directed set. When $(x_k)_{k\in K}$ is a net in X and $a \in X$, we say that $(x_k)_{k\geq K}$ **converges** to a (in X), and we write $x_k \to a$ (in X), when for every open set $U \subseteq X$ with $a \in U$ there exists $m \in K$ such that for all $k \in K$, if $k \geq m$ then $x_k \in U$.

4.33 Theorem: *In a Hausdorff topological space, the limit of a convergent net is unique.*

Proof: The proof is left as an exercise.

4.34 Theorem: *Let X be a topological space, let $A \subseteq X$, and let $a \in X$. Then $a \in \overline{A}$ if and only if there is a net $(x_k)_{k\in K}$ in A with $x_k \to a$ in X.*

Proof: Let $\mathcal{B}$ be any basis for the topology on X (for example, we could let $\mathcal{B}$ be the topology on X) and let $\mathcal{B}_a = \left\{U \in \mathcal{B} \,\middle|\, a \in U\right\}$ (the set of basic open neighbourhoods of a).

Suppose that $a \in \overline{A}$. Note that, by Property 2 in the definition of a basis, $\mathcal{B}_a$ is a directed set under reverse inclusion (that is $U \leq V \iff V \subseteq U$). By Theorem 4.4, since $a \in \overline{A}$ we have $A \cap U \neq \emptyset$ for every $U \in \mathcal{B}_a$, so we can choose an element $x_U \in A \cap U$ for every $U \in \mathcal{B}_a$ to obtain a net $(x_U)_{U\in\mathcal{B}_a}$ in A. Then we have $x_U \to a$ in X because for every open set W in X with $a \in W$ we can choose a basic open set $U \in \mathcal{B}_a$ with $U \subseteq W$, and then for all $V \in \mathcal{B}_a$ with $V \geq U$ we have $x_V \in V \subseteq U \subseteq W$.

Suppose, conversely, that there is a net $(x_k)_{k\in K}$ in A with $x_k \to a$ in X. Then for every basic open set $U \in \mathcal{B}_a$ we can choose $k \in K$ with $x_k \in U$, and so we have $A \cap U \neq \emptyset$. Thus $a \in \overline{A}$ by Theorem 4.4.

4.35 Theorem: *Let X and Y be topological spaces, let $A \subseteq X$, and let $f : A \subseteq X \to Y$. Then f is continuous on A (using the subspace topology in X) if and only if for every $a \in A$ and every net $(x_k)_{k\in K}$ in A, if $x_k \to a$ in X then $f(x_k) \to f(a)$ in Y.*

Proof: Suppose f is continuous on A. Let $a \in A$ and let $(x_k)_{k\in K}$ be a net in A with $x_k \to a$ in X. Let $V \subseteq Y$ be open with $f(a) \in V$. Since f is continuous on A, $f^{-1}(V)$ is open in A. Choose an open set $U \subseteq X$ such that $f^{-1}(V) = U \cap A$. Since $x_k \to a$ in X, we can choose $m \in K$ so that $k \geq m \Longrightarrow x_k \in U$. Then when $k \geq m$ we have $x_k \in U \cap A = f^{-1}(V)$ so that $f(x_k) \in V$. This shows that $f(x_k) \to f(a)$ in Y, as required.

Suppose, conversely, that f is not continuous on A. Choose an open set $V \subseteq Y$ such that $f^{-1}(V)$ is not open in A. Then the set $B = A \setminus f^{-1}(V)$ is not closed in A, so we have

$$B \subsetneq \operatorname{Cl}_A(B) \subseteq \operatorname{Cl}_X(B) = \overline{B}.$$

Choose an element $a \in \operatorname{Cl}_A(B) \setminus B$. Since $a \in \operatorname{Cl}_A(B) \subseteq A$ and $a \notin B = A \setminus f^{-1}(V)$, we have $a \in f^{-1}(V)$ so that $f(a) \in V$. Since $a \in \overline{B}$, by Theorem 4.34 we can choose a net $(x_k)_{k\in K}$ in B with $x_k \to a$ in X. Note that for each $k \in K$, since $x_k \in B = A \setminus f^{-1}(V)$ we have $x_k \notin f^{-1}(V)$ so that $f(x_k) \notin V$. Since V is open in Y, its complement $V^c = Y \setminus V$ is closed in Y so that $\overline{V^c} = V^c$ in Y. Since $\big(f(x_k)\big)_{k\in K}$ is a net in V^c and $f(a) \notin V^c = \overline{V^c}$, it follows from Theorem 4.34 that $f(x_k) \not\to f(a)$ in Y.

4.36 Definition: Let Y be a topological space and let $(f_k)_{k\in K}$ be an indexed set of functions $f_k : X_k \to Y$ where each X_k is a topological space. The **final topology** (or the **srong topology**) on Y (with respect to the indexed set $(f_k)_{k\in K}$) is the finest topology on Y such that each of the functions f_k is continuous. A subset $U \subseteq Y$ is open in the strong topology if and only if ${f_k}^{-1}(V)$ is open in X_k for every open set $V \subseteq X$ and every $k \in K$.

4.37 Example: When X is a topological space and $\sim$ is an equivalence relation on X and $q : X \to X/\!\sim$ is the quotient map given by $q(a) = [a] = \{x \in X \mid x \sim a\}$, the quotient topology on $X/\!\sim$ is equal to the final topology with respect to the quotient map (so the indexed set of maps consists of a single map).

4.38 Definition: Let X be a topological space and let $(f_k)_{k\in K}$ be an indexed set of functions $f_k : X \to Y_k$ where each Y_k is a topological space. The **initial topology** (or the **weak topology**) on X (with respect to the indexed set $(f_k)_{k\in K}$) is the coarsest topology on X such that each of the functions f_k is continuous, that is the topology on X generated by the set $\{{f_k}^{-1}(U) \mid k \in K, U \in Y_k\}$.

4.39 Example: When Y is a topological space and $X \subseteq Y$, the subspace topology on X is equal to the initial topology on X with respect to the inclusion map.

4.40 Example: When X_k is a topological space for each $k \in K$, the product topology on the cartesian product $\prod_{j\in K} X_j$ is equal to the initial topology with respect to $(p_k)_{k\geq K}$, where $p_k : \prod_{j\in K} X_j \to X_k$ is the projection map given by $p_k(x) = x_k$.

4.41 Definition: Let U be a normed linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The **weak topology** on U is the initial topology on U with respect to U^* (that is with respect to the indexed set $(f)_{f\in U^*}$), that is the topology generated by the sets of the form $f^{-1}(V)$ where $f \in U^*$ and V is an open set in $\mathbb{F}$.

The **weak-star** topology (written as the **weak*** topology) on U^* is the initial topology on U^* with respect to the indexed set $(F_u)_{u\in U}$ where $F_u \in U^{**}$ is given by $F_u(f) = f(u)$, that is the topology generated by the sets of the form ${F_u}^{-1}(V) = \{f \in U^* \mid f(u) \in V\}$ where $u \in U$ and V is an open set in $\mathbb{F}$.

4.42 Theorem: *Let U be a normed linear space.*

(1) For every $a \in U$ and every net $(x_k)_{k\in K}$ in U, we have $x_k \to a$ in U using the weak topology if and only if $f(x_k) \to f(a)$ in $\mathbb{F}$ for every $f \in U^*$.
(2) For every $g \in U^*$ and every net $(f_k)_{k\in K}$ in U^*, we have $f_k \to g$ in U^* using the weak* topology if and only if $f_k(x) \to g(x)$ in $\mathbb{F}$ for every $x \in U$.

Proof: We prove Part 1 (the proof of Part 2 is similar). Let $a \in U$ and let $(x_k)_{k\in K}$ be a net in U, and suppose that $x_k \to a$ in U, using the weak toplogy. For every $f \in U^*$, since $x_k \to a$ in U using the weak topology, and since $f : U \to \mathbb{F}$ is continuous when U is using the weak topology, it follows (from Theorem 4.35) that $f(x_k) \to f(a)$ in $\mathbb{F}$. Suppose, conversely, that $f(x_k) \to f(a)$ in $\mathbb{F}$ for every $f \in U^*$. Let $W \subseteq U$ be open in U, using the weak topology, with $a \in W$. Since the weak topology on U is generated by sets of the form $f^{-1}(V)$ with $f \in U^*$ and V open in $\mathbb{F}$, it follows that W is an arbitrary union of finite intersections of elements of this form. Since $a \in W$, a is contained in a finite intersection of elements of this form, say $a \in {f_1}^{-1}(V_1) \cap \cdots \cap {f_n}^{-1}(V_n)$. For each j, since $f_j(x_k) \to f_j(a)$ in $\mathbb{F}$ and $a \in {f_j}^{-1}(V_j)$ so that $f_j(a) \in V_j$, we can choose $m_j \in K$ so that $k \geq m_j \Longrightarrow f_j(x_k) \subseteq V_j \Longrightarrow x_k \in {f_j}^{-1}(V_j)$. We then choose $m \in K$ with $m \geq m_j$ for all j, and then $k \geq m \Longrightarrow x_k \in \bigcap {f_j}^{-1}(V_j) \subseteq W$. Thus $x_k \to a$ in U, as required.

4.43 Remark: Note that when U is an infinite-dimensional normed linear space, and U^* is an infinite dimensional Banach space using the operator norm, the closed unit ball $\overline{B}_{U^*}(0,1) = \big\{ f\in U^* \,\big|\, \|f\| \leq 1 \big\}$ is *not* compact in U^* by Riesz's Theorem (Theorem 3.8).

4.44 Theorem: *(The Banach-Alaoglu Theorem) For a normed linear space U, the closed unit ball $\overline{B}_{U^*}(0,1) = \big\{ f\in U^* \,\big|\, \|f\| \leq 1 \big\}$ is compact in U^* using the weak* topology.*

Proof: Let $B = \overline{B}_U(0,1) = \big\{ x\in U \,\big|\, \|x\| \leq 1 \big\}$ and let $B^* = \overline{B}_{U^*}(0,1) = \big\{ f\in U^* \,\big|\, \|f\| \leq 1 \big\}$. Let $D = \overline{B}_{\mathbb{F}}(0,1) = \big\{ t\in\mathbb{F} \,\big|\, \|t\| \leq 1 \big\}$ and let $P = D^B = \prod_{u\in B} D$ using the product topology. Let $R: B^* \to P$ be the restriction map (an element $f \in B^*$ is a linear map $f: U \to \mathbb{F}$ with $\|f\| \leq 1$, and $R(f)$ is the restriction of f to B, that is $R(f)(x) = f(x)$ for $x \in B$). Note that when $f \in B^*$, the restriction $R(f)$ is in fact an element of P because when $x \in B$ we have $\|f\| \leq 1$ and $\|x\| \leq 1$ so that $\big|f(x)\big| \leq \|f\|\,\|x\| \leq 1$ hence $R(f)(x) = f(x) \in D$, and so $R(f): B \to D$ (and $P = D^B$ is the set of all functions from B to D).

Note that R is injective because given $f, g \in B^*$, if $R(f) = R(g)$ then $f(x) = g(x)$ for all $x \in B$ (that is for all $x \in U$ with $\|x\| \leq 1$) and hence $f(x) = g(x)$ for all $x \in U$ (because f and g are linear) so that $f = g$.

We claim that R is continuous. Recall that a map from a topological space to a cartesian product (using the product topology) is continuous if and only if each of its component functions is continuous, so it suffice to show that R_u is continuous for all $u \in B$, where $R_u : B^* \to D$ is given by $R_u(f) = R(f)_u = R(f)(u) = f(u)$. Let $u \in B$. To show that $R_u : B^* \subseteq U^* \to D \subseteq \mathbb{F}$ is continuous, we shall use Theorem 4.35 (the characterization of continuity by nets). Let $(f_k)_{k\in K}$ be a net in B^*, let $g \in B^*$, and suppose $f_k \to g$ in B^* using the weak* topology. Then we have $f_k(x) \to g(x)$ in $\mathbb{F}$ for all $x \in U$, and hence $R_u(f_k) = f_k(u) \to g(u) = R_u(g)$ for all $u \in B$. Thus R_u is continuous.

We claim that $R(B^*)$ is closed in P. Let $p \in \overline{R(B^*)}$. We need to show that $p \in R(B^*$. By Theorem 4.34 (the characterization of closure by nets) we can choose a net in $R(B^*)$ which converges to p in P, so we can choose a net $(f_k)_{k\in K}$ in B^* such that $R(f_k) \to p$ in P. Since each coordinate projection on P is continuous, we have $R(f_k)(u) \to p(u)$ in D, that is $f_k(u) \to p(u)$ in D, for each $u \in B$. Since each $f_k : U \to \mathbb{F}$ is linear, it follows that the map $p : B \to D \subseteq \mathbb{F}$ is locally linear, meaning that for all $x, y \in U$ and all $t \in \mathbb{F}$, if $x, y, x+y \in B$ then $p(x+y) = p(x) + p(y)$ and if $x, tx \in B$ then $p(tx) = tx$. Since $p : B \to \mathbb{F}$ is locally linear, we can extend p (uniquely) to a linear map $g : U \to \mathbb{F}$ $\big($given $x \in U$ we choose $0 \neq r \in \mathbb{F}$ so that $rx \in B$ and define $g(x) = \frac{1}{r} g(rx)\big)$. Since the restriction of g to B is equal to the map p, and $p : B \to D$, we have $\|g\| \leq 1$ so that $g \in B^*$, and we have $R(g) = p$ so that $p \in R(B^*)$, as required.

Since $R : B^* \to P$ is injective, it gives a bijective map $R : B^* \to R(B^*)$. We claim that the inverse map $R^{-1} : R(B^*) \to B^*$ is continuous. Let $(q_k)_{k\in K}$ be a net in $R(B^*)$ and let $p \in R(B^*)$ with $q_k \to p$ in $R(B^*)$. Let $f_k = R^{-1}(q_k) \in B^*$ so that $q_k = R(f_k)$. Then we have $R(f_k) \to p$ in $R(B^*) \subseteq P$. As above, we have $f_k(u) \to p(u)$ for all $u \in B$, and $p : B \to \mathbb{F}$ extends (uniquely) to a linear map $g : U \to \mathbb{F}$, and then $g \in B^*$ and we have $R(g) = p$ so that $p = R^{-1}(g)$. Since $f_k, g : U \to \mathbb{F}$ are linear and $f_k(u) \to g(u)$ for all $u \in B$, it follows that $f_k(x) \to g(x)$ for all $x \in U$. Since $f_k, g \in B^* \subseteq U^*$ and $f_k(x) \to g(x)$ for all $x \in U$, it follows that $f_k \to g$ in U^* using the weak* topology, that is $R^{-1}(q_k) \to R^{-1}(p)$ in U^* using the weak* topology. Thus R^{-1} is continuous, as claimed.

Since P is compact by Tychonoff's Theorem and $R(B^*)$ is closed in P, it follows that $R(B^*)$ is compact. Since $R : B^* \to R(B^*)$ is a homeomorphism, B^* is also compact.

Locally Convex Topological Vector Spaces

4.45 Definition: A **topological vector space** over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is a vector space with a Hausdorff topology such that the product and sum maps $p : \mathbb{F}\times U \to U$ and $s : U\times U \to U$, given by $p(t, x) = tx$ and $s(x, y) = x + y$, are both continuous (where $\mathbb{F} \times U$ and $U \times U$ use the product topology). When U is a topological vector space, the **linear dual** of U and the **continuous dual** of U are the spaces

$$U^{\#} = \big\{ f : U \to \mathbb{F} \,\big|\, f \text{ is linear} \big\},$$
$$U^{*} = \big\{ f : U \to \mathbb{F} \,\big|\, f \text{ is linear and continuous} \big\}.$$

A topological vector space is said to be **locally convex** when its topology has a basis which consists of convex sets.

4.46 Example: When U is a normed linear space, U, (U, wk) and (U^*, wk^*) are locally convex topological vector spaces. The metric topology on U has a basis consisting of open balls, which are convex, and it is Hausdorff as are all metric spaces. The weak topology on U is generated by sets of the form $f^{-1}(V)$ where $f \in U^*$ and V is an open ball in $\mathbb{F}$, and these sets are convex. A basis for the weak topology is given by the set of finite intersections of such sets $f^{-1}(V)$, and all such finite intersections are convex. To see that the weak topology is Hausdorff, let $u, v \in U$ with $u \neq v$. Define $f : \mathrm{Span}\,\{v - u\} \to \mathbb{F}$ by $f\big(t(v - u)\big) = t$. By the Hahn-Banach Theorem we can extend f to obtain a continuous linear map $f \in U^*$ with $f(v) - f(u) = f(v - u) = 1$. Then the sets $U = f^{-1}\big(B\big(f(u), \frac{1}{2}\big)\big)$ and $V = f^{-1}\big(B\big(f(v), \frac{1}{2}\big)\big)$ are disjoint basic open sets in (U, wk) with $u \in U$ and $v \in V$. We leave it as an exercise to verify that (U^*, wk^*) is locally convex.

4.47 Note: Let X and Y be topological spaces and let $a \in X$ and $b \in Y$. Using the product topology in $X \times Y$, the inclusion maps $j : X \to X \times Y$ and $k : Y \to X \times Y$ given by $j(x) = (x, b)$ and $k(y) = (a, y)$ are continuous.

Proof: We show that j is continuous (the proof that k is continuous is the same). Let $V \subseteq X \times Y$ be open (in the product topology). For each $p \in V$, choose open sets $I_p \subseteq X$ and $J_p \in Y$ such that $p \in I_p \times J_p \subseteq V$. Then $V = \bigcup_{p\in V} I_p \times J_p$ so

$$j^{-1}(V) = \bigcup_{p\in V} j^{-1}\big(I_p \times J_p\big) = \bigcup_{p\in V} \big\{x \in X \,\big|\, x \in I_p, b \in J_p\big\} = \bigcup_{p\in V, b\in J_p} I_p$$

which is open in X, so j is continuous, as required.

4.48 Note: Let U be a topological vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let $a \in U$ and let $0 \neq r \in \mathbb{F}$. The translation $\tau_a : U \to U$ given by $\tau_a(x) = x + a$ and the scaling $\sigma_r : U \to U$ given by $\sigma_r(x) = rx$ are homeomorphisms.

Proof: The translation $\tau_a : U \to U$ is the composite $\tau_a = s \circ j$, where $j : U \to \mathbb{F} \times U$ is the inclusion $j(x) = (x, a)$ and $s : U \times U \to U$ is the summation map $s(x, y) = x + y$, and so every translation τ_a is continuous, and the inverse of the translation τ_a is the translation τ_{-a}, which is also continuous. Similarly, the scaling map $\sigma_r : U \to U$ is the composite $\sigma_r = p \circ k$ where $k : U \to \mathbb{F} \times U$ is the inclusion $k(x) = (r, x)$ and $p : \mathbb{F} \times U \to U$ is the product map $p(t, x) = tx$, so every scaling map σ_r with $r \in \mathbb{F}$ is continuous, and when $r \neq 0$, σ_r is invertible and the inverse of σ_r is the scaling map $\sigma_{1/r}$, which is continuous.

4.49 Note: When U is a real topological vector space and $A \subseteq U$, we have $A^o \subseteq \text{Core}(A)$.

Proof: Let $a \in A^o$ and choose an open set V in U with $a \in V \subseteq A$. Recall that $a \in \text{Core}(A)$ when for every $u \in U$ there exists $r > 0$ such that $a + tu \in A$ for all $t \in (-r, r)$. Let $u \in U$. Since the inclusion map $j : \mathbb{R} \to \mathbb{R} \times U$ given by $j(t) = (t, u)$ is continuous, and the product map $p : \mathbb{R} \times U \to U$ given by $p(t, x) = tx$ is continuous, the composite $f = p \circ j : \mathbb{R} \to U$, given by $f(t) = tu$, is continuous. Since the inclusion map $k : U \to U \times U$ given by $k(y) = (a, y)$ is continuous, and the summation map $s : U \times U \to U$ given by $s(x, y) = x + y$ is continuous, the composite $g = s \circ k : U \to U$ given by $g(u) = a + u$ is continuous. Thus the composite $h = g \circ f : \mathbb{R} \to U$ given by $h(t) = a + tu$ is continuous. Since $V \subseteq U$ is open and h is continuous, $h^{-1}(V)$ is open in $\mathbb{R}$. Since $g(0) = a \in V$ so that $0 \in g^{-1}(V)$, we can choose $r > 0$ such that $(-r, r) \subseteq g^{-1}(V)$. Then we have $a + tu = g(t) \in V \subseteq A$ for all $t \in (-r, r)$, and so $a \in \text{Core}(A)$, as required.

4.50 Theorem: *(Hahn-Banach Separation Theorem for Real Topological Vector Spaces) Let U be a topological vector space over $\mathbb{R}$ and let $\emptyset \neq A, B \subseteq U$ be disjoint convex subsets.*
(1) If A is open then there exists $0 \neq f \in U^*$ and $c \in \mathbb{R}$ such that $f(x) < c \leq f(y)$ for all $x \in A$ and $y \in B$.
(2) If U is locally convex and A is compact and B is closed then there exists $0 \neq f \in U^*$ and $c \in \mathbb{R}$ such that $f(x) < c < f(y)$ for all $x \in A$ and $y \in B$.

Proof: To prove Part 1, suppose that A is open. As in the proof of the Hahn-Banach Separation Theorem (Theorem 3.20), let $a \in A$, let $b \in B$ and let $C = A - B - a + b$. Note that C is convex (because sums of convex sets are convex) and C is open (because C is the union of the open sets $A - y - a + b$ with $y \in B$) and $0 \in C$ and $b - a \notin C$ (because A and B are disjoint so that $0 \notin A - B$). Note that $0 \in C^o \subseteq \text{Core}(C)$ by the above note. Let p be the Minkowski functional of C, given by $p(x) = \inf\left\{r > 0 \,\middle|\, \frac{1}{r}\, x \in C\right\}$, and recall that $p(x) \leq 1$ for all $x \in C$ and $p(b-a) \geq 1$. Let $f : U \to \mathbb{R}$ be the linear map constructed in the proof of Theorem 3.20 with $f(b-a) = p(b-a)$ and $f(x) \leq p(x)$ for all $x \in U$, and recall that $f(x) \leq f(y)$ for all $x \in A$ and $y \in B$. Let $c = \sup\left\{f(x) \,\middle|\, x \in A\right\}$ so that $f(x) \leq c \leq f(y)$ for all $x \in A$ and $y \in B$.

We claim that the map $f : U \to \mathbb{R}$ is continuous (so that $f \in U^*$). Let $V \subseteq \mathbb{R}$ be open. Let $a \in f^{-1}(V)$. Chose $r > 0$ so that $\overline{B}\big(f(a), r\big) \subseteq V$. The set $C \cap -C$ is open in U with $0 \in C \cap -C$. By translating and scaling, the set $a + r(C \cap -C)$ is open in U with $a \in a + r(C \cap -C)$. For all $x \in a + r(C \cap -C)$, we have $\frac{1}{r}(x - a) \in C \cap -C$, so that $\pm\frac{1}{r}(x - a) \in C$, and hence $\pm f\big(\frac{1}{r}(x - a)\big) = f\big(\pm\frac{1}{r}(x - a)\big) \leq p\big(\pm\frac{1}{r}(x - a)\big) \leq 1$ so that $\left|\frac{1}{r} f(x-a)\right| \leq 1$, and hence $\left|f(x) - f(a)\right| \leq r$ so that $f(x) \in \overline{B}\big(f(a), r\big) \subseteq V$. Thus $f^{-1}(V)$ is open, and hence f is continuous, as claimed.

We claim that since A is open and $f \neq 0$ and $f(x) \leq c$ for all $x \in A$, we must have $f(x) < c$ for all $x \in A$. Suppose, for a contradiction, that $x \in A$ with $f(x) = c$. Since $f \neq 0$ we can choose $u \in U$ so that $f(u) \neq 0$ and, by replacing u by $-u$ if necessary, we can choose u so that $f(u) > 0$ (to be specific, we can choose $u = b - a$ so that $f(b-a) = p(b-a) \geq 1$). Since A is open, we have $x \in A^o \subseteq \text{Core}(A)$ so we can choose $r > 0$ so that $x + tu \in A$ for all $t \in (-r, r)$. Then we have $x + \frac{r}{2}u \in A$ and $f\big(x + \frac{r}{2}u\big) = f(x) + \frac{r}{2}f(u) > f(x) = c$, giving the desired contradiction. Thus $f(x) < c$ for all $x \in A$, as claimed.

We remark that if B is also open, we have $f(x) < c < f(y)$ for all $x \in A$ and $y \in B$.

To prove Part 2, suppose that U is locally convex and A is compact and B is closed. We claim that there exists an open convex set C with $0 \in C$ such that $A + C \subseteq B^c$. For each $a \in A$, since B^c is open and U is locally convex we can choose an open convex set C_a with $0 \in C_a$ such that $a+2C_a \subseteq B^c$. The set $\left\{a+C_a \,\middle|\, a\in A\right\}$ is an open cover of A, which is compact, so we can choose $a_1, \cdots, a_n \in A$ so that $A \subseteq \bigcup_{k=1}^n (a_k+C_{a_k})$. Let $C = \bigcap_{k=1}^n C_{a_k}$ and note that C is an open convex set with $0 \in C$. For each $a \in A$ we can choose an index k such that $a \in a_k + C_{a_k}$ and then we have $a + C \in a_k + C_{a_k} + C \subseteq a_k + 2C_{a_k} \subseteq B^c$. Since $a+C \subseteq B^c$ for all $a \in A$, we have $A + C \subseteq B^c$, as required. Since $A + C \cap B = \emptyset$ we have $A+\frac{1}{2}C \cap B-\frac{1}{2}C = \emptyset$. Since $A+\frac{1}{2}C$ and $B-\frac{1}{2}C$ are nonempty disjoint open convex sets, it follows from Part 1 (and the final remark at the end of its proof) that there exists $0 \neq f \in U^*$ and $c \in \mathbb{R}$ such that $f(x) < c < f(y)$ for all $x \in A + \frac{1}{2}C$, $y \in B - \frac{1}{2}C$, hence also for all $x \in A$, $y \in B$.

4.51 Theorem: *(The Hahn-Banach Separation Theorem for Topological Vector Spaces) Let U be a topological vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and let $\emptyset \neq A, B \subseteq U$ be nonempty disjoint convex subsets.*
(1) If A is open then there exists $0 \neq g \in U^*$ and $c \in \mathbb{R}$ such that $\text{Re}\big(g(x)\big) < c \leq \text{Re}\big(g(y)\big)$ for all $x\in A$, $y\in B$.
(2) If U is locally convex and A is compact and B is closed then there exists $0 \neq g \in U^*$ and $c \in \mathbb{R}$ such that $\text{Re}\big(g(x)\big) < c < \text{Re}\big(g(y)\big)$ for all $x \in A$ and $y \in B$

Proof: This follows immediately from the real version (Theorem 4.50) because given a continuous real-linear function $f : U \to \mathbb{R}$ we can define $g : U \to \mathbb{C}$ by $g(x) = f(x) - i\,f(ix)$ and then g is a continuous complex-linear function with $\text{Re}\big(g(x)\big) = f(x)$ for all $x \in U$. We leave it as an easy exercise to verify that g is complex-linear, but let us explain why g is continuous. The map $k : U \to U$ given by $k(x) = -if(ix)$ is continuous by Note 4.48 because it is the composite $k = \sigma_{-i} \circ f \circ \sigma_i$ where σ_i and σ_{-i} are scaling maps. The map $h : U \to U \times U$ given by $h(x) = \big(f(x), k(x)\big)$ is continuous by Theorem 4.29 (a function into a product space is continuous when its component maps are continuous). Our map $g : U \to U$ given by $g(x) = f(x) - if(ix)$ is continuous because it is the composite $g = s \circ h$ where $s : U \times U \to U$ is the sum map $s(x, y) = x + y$.

Reflexive Spaces

4.52 Theorem: *Let U be a normed linear space. The map $I : U \to U^{**}$ given by $I(u)(f) = f(u)$ is a norm preserving (hence injective) linear map.*

Proof: It is easy to see that I is linear. Let us show that I is norm-preserving. Let $u \in U$. For all $f \in U^*$ with $\|f\| \leq 1$ we have $\big|I(u)(f)\big| = \big|f(u)\big| \leq \|f\|\,\|u\| \leq \|u\|$, and it follows that $\|I(u)\| \leq \|u\|$. On the other hand, by Corollary 3.13 to the Heine-Banach Theorem, we can choose $f \in U^*$ with $\|f\| = 1$ such that $f(u) = \|u\|$, and then we have $\big|I(u)(f)\big| = \big|f(u)\big| = \|u\|$, and it follows that $\|I(u)\| \geq \|u\|$. Thus I preserves norm.

4.53 Definition: When U is a normed linear space, the injective norm-preserving linear map $I : U \to U^{**}$ given by $I(u)(f) = f(u)$ is called the **canonical map** from U to U^{**}. We say that U is **reflexive** when I is also surjective, so that I is a norm-preserving isomorphism from U to U^{**}.

4.54 Example: Here are a few examples.

(1) Every finite-dimensional normed linear space is reflexive. Indeed the canonical map $I : U \to U^{**}$ is an injective linear map on finite-dimensional vector spaces, and we have $\dim I(U) = \dim U = \dim U^* = \dim U^{**}$ so that I must be bijective.

(2) Every Hilbert space is reflexive. Indeed when H is a Hilbert space, by the Riesz Representation Theorem for Hilbert Spaces and the definition of the inner product on H^* (Theorem 2.41 and Definition 2.42) we have a bijective linear (or conjugate-linear) map $\phi : H \to H^*$ given by $\phi(u)(x) = \langle x, u\rangle$, and a bijective linear (or conjugate-linear) map $\psi : H^* \to H^{**}$ given by $\psi(f)(g) = \langle f, g\rangle = \langle \phi^{-1}(g), \phi^{-1}(f)\rangle$, and then the composite $\psi\phi : H \to H^{**}$ is the bijective linear map given by

$$(\psi\phi)(u)(f) = \psi\big(\phi(u)\big)(f) = \langle f, \phi(u)\rangle = \langle u, \phi^{-1}(f)\rangle = \phi\big(\phi^{-1}(f)\big)(u) = f(u).$$

So the canonical map I is equal to the composite $\psi\phi$, which is bijective.

(3) For $1 < p < \infty$, the space ℓ_p is reflexive. Indeed, given p with $1 < p < \infty$, let q be the conjugate of p so we have $\frac{1}{p} + \frac{1}{q} = 1$. By the Riesz Representation Theorem for the ℓ_p Spaces (Theorem 1.28), we have isomorphisms $\phi : \ell_q \to {\ell_p}^*$ and $\psi : \ell_p \to {\ell_q}^*$ given by

$$\phi(b)(a) = \sum_{k=1}^{\infty} a_k b_k \ \text{ and } \psi(a)(b) = \sum_{k=1}^{\infty} a_k b_k$$

where $a \in \ell_p$ and $b \in \ell_q$, and the isomorphism $\phi^{-1} : {\ell_p}^* \to \ell_q$ gives the isomorphism $(\phi^{-1})^T : {\ell_q}^* \to {\ell_p}^{**}$ given by $(\phi^{-1})^T(g) = g \circ \phi$. The composite $(\phi^{-1})^T\psi : \ell_p \to {\ell_p}^{**}$ is a bijective linear map. For $a \in \ell_p$ and $f \in {\ell_p}^*$, we have

$$\begin{aligned}\big((\phi^{-1})^T\psi\big)(a)(f) &= (\phi^{-1})^T\big(\psi(a)\big)(f) = \big(\psi(a) \circ \phi^{-1}\big)(f) = \psi(a)\big(\phi^{-1}(f)\big) \\ &= \sum_{k=1}^{\infty} a_k \big(\phi^{-1}(f)\big)_k = \phi\big(\phi^{-1}(f)\big)(a) = f(a).\end{aligned}$$

So the canonical map I is equal to the composite $(\phi^{-1})^T\psi$, which is bijective.

(4) ℓ_1 is not reflexive. Indeed we know, from the Riesz Representation Theorem for the ℓ_p spaces (Theorem 1.28), that ${\ell_1}^* \cong \ell_\infty$ hence ${\ell_1}^{**} \cong {\ell_\infty}^*$. If we had $\ell_1 \cong {\ell_1}^{**}$ then ${\ell_1}^{**}$ would be separable (because ℓ_1 is separable) so ${\ell_\infty}^*$ would be separable (because ${\ell_\infty}^* \cong {\ell_1}^{**}$) and hence ℓ_∞ would be separable by Corollary 3.15, but ℓ_∞ is not separable.

4.55 Theorem: *Let U be a locally convex space. Let $I:U\to U^{**}$ be the canonical map.*
(1) $\big(U,wk\big)^*=U^*$.
(2) $\big(U^*,wk^*\big)^*=I(U)$.

Proof: To prove Part 1, let $f:U\to\mathbb{F}$ be a linear map. If $f\in U^*$ then f is continuous on $\big(U,\text{wk}\big)$ because the weak topology, by definition, is the coarsest topology on U for which every element $f\in U^*$ is continuous. If $f\in\big(U,\text{wk}\big)^*$, then for every open set $V\subseteq\mathbb{F}$, the set $f^{-1}(V)$ is open in $\big(U,\text{wk}\big)$, so $f^{-1}(V)$ is also open in U (using the norm topology, which is finer than the weak topology), and hence f is continuous on U (using the norm topology).

To prove Part 2, let $\varphi:U^*\to\mathbb{F}$ be a linear map. If $\varphi=I(u)$ for some $u\in U$, then φ is continuous on $\big(U^*,\text{wk}^*\big)$ because the weak* topology, by definition, is the coarsest topology for which every map of the form $I(u)$ with $u\in U$ is continuous. Suppose $\varphi\in\big(U^*,\text{wk}^*\big)^*$. Then $\varphi^{-1}\big(B(0,1)\big)$ is open in $\big(U^*,\text{wk}^*\big)$, and so $\varphi^{-1}\big(B(0,1)\big)$ is an arbitrary union of finite intersections of sets of the form $\big\{f\in U^*\big|f(u)\in V\big\}$ where $u\in U$ and $V\subseteq\mathbb{F}$ is open. In particular, the element $0\in U^*$ lies in one of those finite intersections, so we can choose elements $u_1,\cdots,u_n\in U$ and open sets $V_1,\cdots,V_n\subseteq\mathbb{F}$ (all containing 0) such that $\bigcap_{k=1}^n\big\{f\in U^*\,\big|\,f(u_k)\subseteq V_k\big\}\subseteq\varphi^{-1}\big(B(0,1)\big)$. Choosing $r>0$ small enough so that $B(0,r)\subseteq V_k$ for every k, we have

$$\bigcap_{k=1}^n\big\{f\in U^*\,\big|\,\big|f(u_k)\big|<r\big\}\subseteq\varphi^{-1}\big(B(0,1)\big).$$

Thus for all $f\in U^*$, if $\big|f(u_k)\big|<r$ for all $1\le k\le n$ then $\big|\varphi(f)\big|<1$. It follows that if $f(u_k)=0$ for all $1\le k\le n$ then $\varphi(f)=0$: indeed if $f(u_k)=0$ for all k then for all $n\in\mathbf{Z}^+$ we have $\big|(nf)(u_k)\big|=0$ so that $\big|\varphi(nf)\big|<1$ and hence $\big|\varphi(f)\big|<\frac{1}{n}$. Letting $\varphi_k=I(u_k)$ so that $\varphi_k(f)=f(u_k)$, we have $\bigcap_{k=1}^n\ker(\varphi_n)\subseteq\ker\varphi$, and it follows from linear algebra that $\varphi\in\text{Span}\,\{\varphi_1,\cdots,\varphi_n\}$, say $\varphi=\sum c_k\varphi_k$. Then for all $f\in U^*$ we have $\varphi(f)=\sum c_k\varphi_k(f)=\sum c_kf(u_k)=f\big(\sum c_ku_k\big)$ and hence $\varphi=I(u)$ where $u=\sum c_ku_k$.

4.56 Theorem: *(Goldstine's Theorem) Let U be a normed linear space. Let $I:U\to U^{**}$ be the canonical map. Then $I\big(\overline{B}_U(0,1)\big)$ is dense in $\overline{B}_{U^{**}}(0,1)$ in the space $\big(U^{**},wk^*\big)$.*

Proof: Let $B=\overline{B}_U(0,1)\subseteq U$ and $B^{**}=\overline{B}_{U^{**}}(0,1)\subseteq U^{**}$, and let $\overline{I(B)}$ denote the closure of $I(B)$ in $\big(U^{**},\text{wk}^*\big)$. Let $J:U^*\to U^{***}$ be the canonical map given by $J(f)(\varphi)=\varphi(f)$ where $f\in U^*$ and $\varphi\in U^{**}$. Suppose, for a contradiction, that $B^{**}\setminus\overline{I(B)}\neq\emptyset$ and choose $\varphi\in B^{**}\setminus\overline{I(B)}$. Then $\{\varphi\}$ and $\overline{I(B)}$ are disjoint nonempty convex sets in $\big(U^{**},\text{wk}^*\big)$ with $\{\varphi\}$ compact. By the Hahn-Banach Theorem for Topological Vector Spaces, applied to the locally convex space $\big(U^{**},\text{wk}^*\big)^*=J(U^*)\subseteq U^{***}$, we can choose $g\in U^*$ and $c\in\mathbb{R}$ such that

$$\text{Re}\,\big((Jg)(\psi)\big)<c<\text{Re}\,\big((Jg)(\varphi)\big)\ \text{ for all }\ \psi\in\overline{I(B)}.$$

This implies that $\text{Re}\,\big(\psi(g)\big)<c<\text{Re}\,\big(\varphi(g)\big)$ for all $\psi=I(u)\in I(B)$, that is

$$\text{Re}\,\big(g(u)\big)<c<\text{Re}\,\big(\varphi(g)\big)\ \text{ for all }\ u\in B.$$

In particular, since $0\in B$ we have $0=\text{Re}\,\big(g(0)\big)<c$. Let $h=\frac{1}{c}\,g\in U^*$, so we have

$$\text{Re}\,\big(h(u)\big)<1<\text{Re}\,\big(\varphi(h)\big)\ \text{ for all }\ u\in B.$$

Given $u\in B$ (that is given $u\in U$ with $\|u\|\le1$) we choose $\theta\in\mathbb{R}$ such that $h(u)=|h(u)|e^{i\theta}$ and then we have $\big|h(u)\big|=\text{Re}\,\big(|h(u)|\big)=\text{Re}\,\big(e^{-i\theta}h(u)\big)=\text{Re}\,\big(h(e^{-i\theta}u)\big)<1$. This shows that $\|h\|\le1$. But then we have $1<\text{Re}\,\big(\varphi(h)\big)\le\big|\varphi(h)\big|\le\|\varphi\|\,\|h\|\le1$ which is impossible.

4.57 Theorem: *Let U be a Banach space. Then the following are equivalent:*
(1) U is reflexive.
(2) U^* is reflexive.
(3) The weak topology on U^* is equal to the weak* topology on U^*.
(4) The unit ball $\overline{B}_U(0,1) = \big\{x \in U \,\big|\, \|x\| \leq 1\big\}$ is compact in U using the weak topology.

Proof: Let $B = \overline{B}_U(0,1) \subseteq U$, $B^* = \overline{B}_{U^*}(0,1) \subseteq U^*$ and $B^{**} = \overline{B}_{U^{**}}(0,1) \subseteq U^{**}$, and let $I : U \to U^{**}$ be the canonical embedding given by $I(u)(f) = f(u)$.

To prove that (1) $\Longrightarrow$ (4), suppose that U is reflexive, that is suppose that $I : U \to U^{**}$ is bijective. The weak topology in U is generated by sets of the form $\{u \in U \,|\, f(u) \in V\}$ with $f \in U^*$ and $V \subseteq \mathbb{F}$ open, and the weak* topology in U^{**} is generated by the sets of the form $\big\{\varphi \in U^{**} \big| \varphi(f) \in V\big\}$ with $f \in U^*$ and $V \subseteq \mathbb{F}$ is open. Since every $\varphi \in U^{**}$ is of the form $\varphi = \varphi_u = I(u)$, so $\varphi_u(f) = f(u)$, we have

$$\big\{\varphi \in U^{**} \big| \varphi(f) \in V\big\} = \big\{\varphi_u \in U^{**} \big| f(u) \in V\big\} = I\big(\{u \in U \mid f(u) \in V\}\big)$$

and so the canonical map $I : U \to U^{**}$ is a homeomorphism from (U, wk) to (U^{**}, wk^*). Also note that since $I : U \to U^{**}$ is a norm-preserving isomorphism, we have $IB = B^{**}$. By the Banach-Alaoglu Theorem, $IB = B^{**}$ is compact in $\big(U^{**}, \text{wk}^*\big)$ and hence, since $I : (U, \text{wk}) \to (U^{**}, \text{wk}^*)$ is a homeomorphism, it follows that B is compact in $\big(U, \text{wk}\big)$.

To prove (4) $\Longrightarrow$ (1), suppose that B is compact in (U, wk). The open sets in (U^{**}, wk^*) are generated by sets of the form $\big\{\varphi \in U^{**} \big| \varphi(f(\in V\big\}$ with $f \in U^*$ and $V \subseteq \mathbb{F}$ open. The open sets in the subspace $I(U)$ (using the subspace topology) are generated by the sets of the form

$$\big\{\varphi \in U^{**} \big| \varphi(f) \in V\big\} \cap I(U) = \big\{\varphi_u \in U^{**} \big| f(u) \in V\big\} = I\big(\{u \in U \,\big|\, f(u) \in V\big\}$$

and so the canonical map $I : U \to U^{**}$ is a homeomorphism from (U, wk) to the image $I(U) \subseteq (U^{**}, \text{wk}^*)$ using the subspace topology. Since B is compact in (U, wk), $I(B)$ is compact in $I(U) \subseteq (U^{**}, \text{wk}^*)$ using the subspace topology, and hence $I(B)$ is compact in (U^{**}, wk^*) (by Theorem 4.12), and hence closed in (U^{**}, wk^*) (by Theorem 4.17). Since $I : U \to U^{**}$ is norm-preserving, we have $I(B) \subseteq B^{**}$. Since $I(B) \subseteq B^{**}$ and $I(B)$ is closed in (U^{**}, wk^*) and $I(B)$ is dense in B^{**} in (U^{**}, wk^*) by Goldstine's Theorem, it follows that $I(B) = B^{**}$. Since $I(B) = B^{**}$, the map I is surjective: indeed given $0 \neq \varphi \in U^{**}$ we have $\frac{\varphi}{\|\varphi\|} \in B^{**}$ so we can choose $u \in B$ such that $I(u) = \frac{\varphi}{\|\varphi\|}$ and then we have $I\big(\|\varphi\|\, u\big) = \varphi$.

To prove (1) $\Longrightarrow$ (3), suppose U is reflexive. The weak topology on U^* is generated by the sets of the form $\varphi^{-1}(V)$ with $\varphi \in U^{**}$ and $V \subseteq \mathbb{F}$ open, and the weak* topology on U^* is generated by the sets ${\varphi_u}^{-1}(V)$ with $u \in U$ and $V \subseteq \mathbb{F}$ open, and these are exactly the same generating sets because I is bijective so the elements $\varphi \in U^{**}$ are the same as the elements φ_u with $u \in U$.

To prove that (3) $\Longrightarrow$ (2), suppose that $(U^*, \text{wk}) = (U^*, \text{wk}^*)$. By the Banach-Alaoglu Theorem, B^* is compact in (U^*, wk^*), hence in (U^*, wk). By our proof that (4) $\Longrightarrow$ (1), it follows that U^* is reflexive.

To prove (2) $\Longrightarrow$ (1), suppose that U^* is reflexive. Since $I : U \to U^{**}$ preserves norm, it preserves Cauchy sequences and limits in the norm topology, and so $I(B) \subseteq B^{**}$ is closed in U^{**} (using the norm topology). Since $I(B)$ is convex and closed in U^{**} (in the norm topology), it is closed in (U^{**}, wk) by Question 2(c) on Assignment 4. Since U^* is reflexive, by our proof that (1) $\Longrightarrow$ (3) we have $(U^{**}, \text{wk}) = (U^{**}, \text{wk}^*)$, so $I(B)$ is closed in (U^{**}, wk^*). Since $I(B) \subseteq B^{**}$, $I(B)$ is closed in (U^{**}, wk^*) and $I(B)$ is dense in B^{**} in (U^{**}, wk^*) by Goldstine's Theorem, we have $I(B) = B^{**}$ hence $I(U) = U^{**}$ (as above).