Chapter 1. Preliminaries

Basic Definitions

1.1 Definition: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let U be a vector space over $\mathbb{F}$. An **inner product** on U (over $\mathbb{F}$) is a function $\langle\ ,\ \rangle : U \times U \to U$ (meaning that if $u, v \in U$ then $\langle u, v\rangle \in U$) such that for all $u, v, w \in U$ and all $t \in \mathbb{F}$ we have

(1) (Sesquilinearity) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$, $\langle tu, v\rangle = t\,\langle u, v\rangle$,
$\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$, $\langle u, tv\rangle = \overline{t}\,\langle u, v\rangle$,
(2) (Conjugate Symmetry) $\langle u, v\rangle = \overline{\langle v, u\rangle}$, and
(3) (Positive Definiteness) $\langle u, u\rangle \geq 0$ with $\langle u, u\rangle = 0 \iff u = 0$.

For $u, v \in U$, $\langle u, v\rangle$ is called the inner product of u with v. An **inner product space** (over $\mathbb{F}$) is a vector space over $\mathbb{F}$ equipped with an inner product. Given two inner product spaces U and V over $\mathbb{F}$, a linear map $L : U \to V$ is called a **homomorphism** of inner product spaces (or we say that L **preserves inner product**) when $\big\langle L(x), L(y)\big\rangle = \langle x, y\rangle$ for all $x, y \in U$.

1.2 Definition: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let U be a vector space over $\mathbb{F}$. A **norm** on U is a map $\|\ \| : U \to \mathbb{R}$ such that for all $u, v \in U$ and all $t \in \mathbb{F}$ we have

(1) (Scaling) $\|tu\| = |t|\,\|u\|$,
(2) (Positive Definiteness) $\|u\| \geq 0$ with $\|u\| = 0 \iff u = 0$, and
(3) (Triangle Inequality) $\|u+v\| \leq \|u\| + \|v\|$.

For $u \in U$ the real number $\|u\|$ is called the **norm** (or **length**) of u, and we say that u is a **unit vector** when $\|u\| = 1$. A **normed linear space** over $\mathbb{F}$ is a vector space over $\mathbb{F}$ equipped with a norm. Given two normed linear spaces U and V over $\mathbb{F}$, a linear map $L : U \to V$ is called a **homomorphism** of normed linear spaces (or we say that L **preserves norm**) when $\big\|L(x)\big\| = \|x\|$ for all $x \in U$.

1.3 Theorem: *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let U be an inner product space over $\mathbb{F}$. For $u \in U$ define $\|u\| = \sqrt{\langle u, u\rangle}$. Then*

(1) (Scaling) $\|tu\| = |t|\,\|u\|$,
(2) (Positive Definiteness) $\|u\| \geq 0$ with $\|u\| = 0 \iff u = 0$,
(3) $\|u+v\|^2 = \|u\|^2 + 2\,\mathrm{Re}\,\langle u, v\rangle + \|v\|^2$,
(4) (Pythagoras' Theorem) if $\langle u, v\rangle = 0$ then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$,
(5) (Parallelogram Law) $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$,
(6) (Polarization Identity) if $\mathbb{F} = \mathbb{R}$ then $\langle u, v\rangle = \frac{1}{4}\big(\|u+v\| - \|u-v\|\big)$ and if $\mathbb{F} = \mathbb{C}$ then $\langle u, v\rangle = \frac{1}{4}\big(\|u+v\|^2 + i\|u+iv\|^2 - \|u-v\|^2 - i\|u-iv\|^2\big)$,
(7) (The Cauchy-Schwarz Inequality) $|\langle u, v\rangle| \leq \|u\|\|v\|$ with $|\langle u, v\rangle| = \|u\|\|v\|$ if and only if $\{u, v\}$ is linearly dependent, and
(8) (The Triangle Inequality) $\|u+v\| \leq \|u\| + \|v\|$.

In particular, $\|\ \|$ is a norm on U.

Proof: We omit the proof.

1.4 Definition: A **metric** on a nonempty set X is a function $d : X \times X \to \mathbb{R}$ such that, for all $x, y, z \in X$ we have

(1) (Positive Definiteness) $d(x,y) \geq 0$ with $d(x,y) = 0 \iff x = y$,
(2) (Symmetry) $d(x,y) = d(y,x)$ and
(3) (Triangle Inequality) $d(x,z) \leq d(x,y) + d(y,z)$.

A nonempty set with a metric is called a **metric space**.

1.5 Definition: A **topology** on a set X is a set $\mathcal{T}$ of subsets of X such that

(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
(2) if $U \in \mathcal{T}$ and $V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$, and
(3) if K is a set and $U_k \in \mathcal{T}$ for each $k \in K$ then $\bigcup_{k \in K} U_k \in \mathcal{T}$.

For a subset $A \subseteq X$, we say that A is **open** (in X) when $A \in \mathcal{T}$ and we say that A is **closed** (in X) when $X \setminus A \in \mathcal{T}$. A set with a topology is called a **topological space**.

1.6 Note: Given an inner product on a vector space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, Theorem 1.3 shows that we can define an associated norm on V by letting $\|x\| = \sqrt{\langle x, x \rangle}$ for $x \in V$.

Given a norm on a vector space V, we can define an associated metric on any subset $X \subseteq V$ by letting $d(x,y) = \|x - y\|$ for $x, y \in X$.

Given a metric on a set X, recall (or verify) that we can define an associated topology on X by stipulating that a subset $A \subseteq X$ is open when it has the property that for all $a \in A$ there exists $r > 0$ such that $B(a,r) \subseteq A$, where $B(a,r) = \{x \in X \,|\, d(x,a) < r\}$.

1.7 Definition: Let $(x_n)_{n \geq 1}$ be a sequence in a metric space X. For $a \in X$, we say that the sequence (x_n) **converges** to a in X, and we write $\lim_{n \to \infty} x_n = a$, or we write $x_n \to a$, when

$$\forall \epsilon > 0 \; \exists n \in \mathbf{Z}^+ \; \forall k \in \mathbf{Z}^+ \; \big(k \geq n \Longrightarrow d(x_n, a) < \epsilon\big).$$

We say that (x_n) **converges** in X when it converges to some element $a \in X$. We say that (x_n) is **Cauchy** when

$$\forall \epsilon > 0 \; \exists n \in \mathbf{Z}^+ \; \forall k, l \in \mathbf{Z}^+ \; \big(k, l \geq n \Longrightarrow d(x_k, x_l) < \epsilon\big).$$

Recall (or verify) that, in a metric space, if a sequence converges then it is Cauchy.

1.8 Definition: A metric space X is called **complete** when, in X, every Cauchy sequence converges. Note that if X is complete and $A \subseteq X$ is closed then A is also complete. A complete normed linear space is called a **Banach space** and a complete inner-product space is called a **Hilbert space**.

1.9 Definition: Let X be a topological space and let $A \subseteq X$. We say that A is **dense** in X when $\overline{A} = X$ (where $\overline{A}$ denotes the closure of A in X). In the case that X is a metric space, A is closed when for every sequence (x_n) in A and every $a \in X$, if $x_n \to a$ in X then $a \in A$. A metric space is called **separable** when it contains a countable dense subset.

Examples

1.10 Example: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The **standard inner product** of $\mathbb{F}^n$ is given by $\langle x, y\rangle = y^* x = \sum_{k=1}^{n} x_k \overline{y_k}$. It induces the **standard norm**, also called the **2-norm**, on $\mathbb{F}^n$ given by $\|x\| = \|x\|_2 = \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2}$ and the **standard metric**, also called the **2-metric**, on $\mathbb{F}^n$ given by $d(x,y) = d_2(x,y) = \|x-y\|_2$. Recall that $\mathbb{F}^n$ is complete and separable under d_2, so it is a finite-dimensional separable Hilbert space.

1.11 Example: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let $\mathbb{F}^\omega$ be the space of sequences $x = (x_1, x_2, x_3, \cdots)$ with each $x_k \in \mathbb{F}$, and let $\mathbb{F}^\infty$ be the space of eventually zero sequences ($\mathbb{F}^\infty$ is countable-dimensional with basis $\{e_1, e_2, e_3, \cdots\}$ where $e_1 = (1, 0, 0\cdots)$, $e_2 = (0, 1, 0, \cdots)$ and so on). Let

$$\ell_2 = \ell_2(\mathbb{F}) = \left\{x \in \mathbb{F}^\omega \,\Big|\, \sum_{k=1}^{n} |x_k|^2 < \infty\right\}.$$

Recall that we can define an inner product, called the **standard inner product**, on ℓ_2 given by $\langle x, y\rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}$. It induces the **2-norm** given by $\|x\|_2 = \left(\sum_{k=1}^{\infty} |x_k|\right)^{1/2}$, and the **2-metric** given by $d_2(x,y) = \|x-y\|_2$. Recall that ℓ_2 is complete and separable under d_2, so it is an infinite-dimensional separable Hilbert space.

1.12 Example: Let $A \subseteq \mathbb{R}$ be measurable and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Recall that when $u, v : A \to \mathbb{R}$, the function $f = u + iv : A \to \mathbb{C}$ is measurable if and only if u and v are both measurable and, in this case, $\int_A f = \int_A u + i \int_A v$. Let $\mathcal{M}(A) = \mathcal{M}(A, \mathbb{F})$ denote the set of all measurable functions $f : A \to \mathbb{F}$, and let

$$L_2(A) = L_2(A, \mathbb{F}) = \left\{f \in \mathcal{M}(A) \Big| \int_A |f|^2 < \infty\right\} \Big/ \sim$$

where $\sim$ is the equivalence relation given by $f \sim g \iff f = g$ a.e. in A. Recall that we can define an inner product, called the **standard inner product**, on $L_2(A)$ by $\langle f, g\rangle = \int_A f\overline{g}$. It induces the **2-norm** given by $\|f\|_2 = \left(\int_A |f|^2\right)^{1/2}$ and the **2-metric** given by $d_2(f,g) = \|f-g\|_2$. Recall that $L_2(A)$ is complete under d_2. Also recall that for $a < b$, $L_2[a,b]$ is separable so it is an infinite-dimensional separable Hilbert space.

1.13 Example: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $x \in \mathbb{F}^n$, define $\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}$ for $1 \le p < \infty$, and $\|x\|_\infty = \max\left\{|x_k| \,\big|\, 1 \le k \le n\right\}$. Recall that for $1 \le p \le \infty$, $\|x\|_p$ gives a norm, called the **p-norm** on $\mathbb{F}^n$, and it induces the **p-metric** d_p. The ∞-norm is also called the **supremum norm** and the ∞-metric is also called the **supremum metric**. Recall that $\mathbb{F}^n$ is complete and separable under d_p for $1 \le p \le \infty$, so it is a finite-dimensional separable Banach space.

1.14 Example: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $x \in \mathbb{F}^\omega$, define $\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}$ for $1 \le p < \infty$, and $\|x\|_\infty = \sup\left\{|x_k| \,\big|\, k \in \mathbf{Z}^+\right\}$ (note that $\|x\|_p$ and $\|x\|_\infty$ can be infinite), and let

$$\ell_p = \ell_p(\mathbb{F}) = \left\{x \in \mathbb{F}^\omega \,\big|\, \|x\|_p < \infty\right\} \quad \text{and} \quad \ell_\infty = \ell_\infty(\mathbb{F}) = \left\{x \in \mathbb{F}^\omega \,\big|\, \|x\|_\infty < \infty\right\}.$$

Recall that for $1 \le p \le \infty$, $\|x\|_p$ gives a norm, called the **p-norm** on ℓ_p, and it induces the **p-metric** d_p. The ∞-norm is also called the **supremum norm** and the ∞-metric is also called the **supremum metric**. Recall that ℓ_p is complete under d_p for $1 \le p \le \infty$, so it is a Banach space. Also recall that ℓ_p is separable for $1 \le p < \infty$, but ℓ_∞ is not separable.

1.15 Example: Let $A \subseteq \mathbb{R}$ be measurable and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{M}(A) = \mathcal{M}(A, \mathbb{F})$ be the set of all (Lebesgue) measurable functions $f : A \to \mathbb{F}$. When $f \in \mathcal{M}(A)$, for $1 \leq p < \infty$, the **p-norm** of f is given by

$$\|f\|_p = \left(\int_A |f|^p \right)^{1/p}$$

and the **∞-norm**, also called the **essential supremum**, of f is given by

$$\|f\|_\infty = \inf \left\{ m \geq 0 \,\middle|\, |f(x)| \leq m \text{ a.e. on } A \right\}$$

(note that $\|f\|_p$ and $\|f\|_\infty$ can by infinite). Let

$$\begin{aligned} L_p(A) &= \left\{ f \in \mathcal{M}(A) \,\middle|\, \|f\|_p < \infty \right\} \Big/ \sim \quad \text{for } 1 \leq p < \infty, \text{ and} \\ L_\infty(A) &= \left\{ f \in \mathcal{M}(A) \,\middle|\, \|f\|_\infty < \infty \right\} \Big/ \sim \end{aligned}$$

where $\sim$ is the equivalence relation given by $f \sim g \iff f = g$ a.e. in A. Recall that the p-norm is indeed a norm on $L_p(A)$ for $1 \leq p \leq \infty$, and it induces the **p-metric** d_p. The ∞-norm is also called the **supremum norm** and the ∞-metric is also called the **supremum metric**. Recall that $L_p(A)$ is complete under d_p for $1 \leq p \leq \infty$, so it is a Banach space. Also recall that for $a < b$, $L_p[a, b]$ is separable for $1 \leq p < \infty$ but $L_\infty[a, b]$ is not separable.

1.16 Remark: In Examples 1.11 and 1.14, we need to quotient by the equivalence relation $\sim$ in order that the inner product and the p-norms are positive definite. In Examples 1.13, 1.14 and 1.15, it is not immediately obvious, from the definition of the various p-norms, that they satisfy the Triangle Inequality. The Triangle Inequality for the p-norms is called Minkowski's Inequality, and it is often proven using Hölder's Inequality.

1.17 Theorem: *(Hölder's Inequality) Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ and let $A \subseteq \mathbb{R}$ be measurable.*
(1) For all $x, y \in \mathbb{F}^n$ and for all $x, y \in \mathbb{F}^\omega$ we have $\|xy\|_1 \leq \|x\|_p \|y\|_q$.
(2) For all $f, g \in \mathcal{M}(A)$ we have $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

Proof: We omit the proof.

1.18 Theorem: *(Minkowski's Inequality) Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let $p \in [1, \infty]$ and let $A \subseteq \mathbb{R}$ be measurable.*
(1) For all $x, y \in \mathbb{F}^n$ and for all $x, y \in \mathbb{F}^\omega$ we have $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.
(2) For all $f, g \in \mathcal{M}(A)$ we have $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Proof: We omit the proof.

1.19 Example: Let X be a metric space and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{F}(X) = \mathcal{F}(X, \mathbb{F})$ be the space of all functions $f : X \to \mathbb{F}$. let $\mathcal{F}_b(X) = \mathcal{F}_b(X, \mathbb{F})$ be the space of bounded functions, let $\mathcal{C}(X) = \mathcal{C}(X, \mathbb{F})$ be the space of continuous functions, and let $\mathcal{C}_b(X) = \mathcal{C}_b(X, \mathbb{F})$ be the space of bounded continuous functions. Recall that $\mathcal{F}_b(X)$ is a Banach space under the **supremum metric** $\|f\|_\infty = \sup \left\{ |f(x)| \,\middle|\, x \in X \right\}$ (indeed, convergence in $\mathcal{F}_b(X)$ under the supremum metric is the same thing as uniform convergence on X, and $\mathcal{F}_b(X)$ is complete because the uniform limit of a sequence of continuous functions is continuous). Also $\mathcal{C}_b(X)$ is closed in $\mathcal{F}_b(X)$, so it is also a Banach space under the supremum norm. When X is compact we have $\mathcal{C}(X) = \mathcal{C}_b(X)$, so $\mathcal{C}(X)$ is a Banach space. When $a, b \in \mathbb{R}$ with $a < b$, $\mathcal{C}[a, b]$ is separable by the Weierstrass Polynmial Approximation Theorem.

Bounded Linear Operators

1.20 Remark: When U and V are normed linear spaces over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, a linear map $F:U\to V$ is also called a **linear transformation** or a **linear operator**, a linear map $F:U\to U$ is also called a **linear operator** on U, and a linear map $F:U\to\mathbb{F}$ is also called a **linear functional** on U. In the past, the words operator and functional were normally used when U was an infinite-dimensional space of functions.

1.21 Definition: Let U and V be normed linear spaces and let $F:U\to V$ be a linear operator. The **operator norm** of F is given by

$$\|F\|=\sup\left\{\|Fx\|\,\Big|\,x\in U\text{ with }\|x\|\le 1\right\}$$

and we say that F is **bounded** when $\|F\|<\infty$, that is when the set $F\big(\overline{B}_U(0,1)\big)$ is bounded in V. Since $Fx=\|x\|\,F\big(\frac{x}{\|x\|}\big)$ for all $0\neq x\in U$, it follows that (when $U\neq\{0\}$)

$$\|F\|=\sup\left\{\|Fx\|\,\Big|\,x\in U\text{ with }\|x\|=1\right\}$$

and that

$$\|Fx\|\le\|F\|\,\|x\|\text{ for all }x\in U$$

with

$$\|F\|=\inf\left\{m\ge 0\,\Big|\,\|Fx\|\le m\|x\|\text{ for all }x\in U\right\}.$$

We denote the **space of bounded linear operators** $F:U\to V$ by $\mathcal{B}(U,V)$, so

$$\mathcal{B}(U,V)=\left\{F:U\to V\,\Big|\,F\text{ is linear with }\|F\|<\infty\right\}.$$

1.22 Example: Recall, from linear algebra, that when U and V are non-trivial finite dimensional inner product spaces over $\mathbb{R}$ and $F:U\to V$ is a linear map, the closed unit ball in U is compact (so that $\|Fx\|$ attains its maximum on the closed unit ball) and we have

$$\|F\|=\max\left\{\|Fx\|\,\Big|\,x\in U,\|x\|=1\right\}=\|Fu\|=\sqrt{\lambda}$$

where λ is the largest eigenvalue of $F^*F:U\to U$ and u is a unit eigenvector for λ.

1.23 Theorem: *Let U and V be normed linear spaces.*
(1) The set $\mathcal{B}(U,V)$ is a normed linear space using the operator norm.
(2) If V is a Banach space then $\mathcal{B}(U,V)$ is a Banach space.

Proof: To prove Part 1, let $F,G:U\to V$ be linear operators. It is clear, from the definition of $\|F\|$ that $\|F\|\ge 0$ with $\|F\|=0\iff F=0$. For all $x\in U$ and $t\in\mathbb{F}$, we have

$$\|(tF)x\|=\|t(Fx)\|=|t|\,\|Fx\|\le|t|\,\|F\|\,\|x\|$$

and it follows that $\|tF\|=|t|\,\|F\|$. Also, for all $x,y\in U$ we have

$$\|(F+G)x\|=\|Fx+Gx\|\le\|Fx\|+\|Gx\|\le\|F\|\,\|x\|+\|G\|\,\|x\|=\big(\|F\|+\|G\|\big)\|x\|$$

and it follows that $\|F+G\|\le\|F\|+\|G\|$. Thus $\mathcal{B}(U,V)$ is a vector space (it is a subspace of the space $\mathrm{Hom}(U,V)$ of linear maps from U to V because $0\in\mathcal{B}(U,V)$ and when F and G are bounded and $t\in\mathbb{F}$, the operators tF and $F+G$ are bounded) and the operator norm is indeed (as its name suggests) a norm on $\mathcal{B}(U,V)$.

To prove Part 2, suppose that V is a Banach space and let $(F_n)_{n\geq 1}$ be a Cauchy sequence in $\mathcal{B}(U,V)$. Note that for each $x\in U$ and for $k,\ell\in\mathbf{Z}^+$ we have

$$\|F_kx-F_\ell x\|=\|(F_k-F_\ell)x\|\leq\|F_k-F_\ell\|\,\|x\|$$

and it follows that $(F_nx)_{n\geq 1}$ is a Cauchy sequence in V and hence, since V is a Banach space, it converges. Define $G:U\to V$ by $Gx=\lim\limits_{n\to\infty}F_nx$. Note that G is linear.

We claim that G is bounded. Since the sequence $(F_n)_{n\geq 1}$ is Cauchy in $\mathcal{B}(U,V)$ it is also bounded in $\mathcal{B}(U,V)$, so we can choose $M\geq 0$ such that $\|F_n\|\leq M$ for all $n\in\mathbf{Z}^+$. Then for all $x\in U$, since $F_nx\to Gx$ in V we have $\|F_nx\|\to\|Gx\|$ in $\mathbb{R}$ (because $\big|\|F_nx\|-\|Gx\|\big|\leq\|F_nx-Gx\|$), and since $\|F_nx\|\leq\|F_n\|\,\|x\|\leq M\,\|x\|$ for all n, we have

$$\|Gx\|=\Big\|\lim_{n\to\infty}F_nx\Big\|=\lim_{n\to\infty}\|F_nx\|\leq M\|x\|.$$

Thus $\|G\|\leq M$ so that G is bounded, as claimed.

Finally, we claim that $F_n\to G$ in $\mathcal{B}(U,V)$. Let $\epsilon>0$. Since $(F_n)_{n\geq 1}$ is Cauchy in $\mathcal{B}(U,V)$, we can choose $m\in\mathbf{Z}^+$ such that when $k,n\geq m$ we have $\|F_n-F_k\|<\epsilon$. Then when $k,n\geq m$, for all $x\in U$ we have

$$\|F_nx-F_kx\|=\|(F_n-F_k)x\|\leq\|F_n-F_k\|\,\|x\|<\epsilon\|x\|.$$

Let $n\geq m$. Then for all $x\in U$ we have

$$\|(F_n-G)x\|=\|F_nx-Gx\|=\Big\|F_nx-\lim_{k\to\infty}F_kx\Big\|=\lim_{k\to\infty}\|F_nx-F_kx\|\leq\epsilon\|x\|.$$

Thus $F_n\to G$ in $\mathcal{B}(U,V)$, as claimed.

1.24 Definition: Let (X,d_X) and (Y,d_Y) be metric spaces, and let $f:X\to Y$. We say that f is **Lipschitz continuous** (on X) when there is a constant $\ell\geq 0$, called a **Lipschitz constant** for f, such that for all $x,y\in X$ we have $d_Y\big(f(x),f(y)\big)\leq\ell\cdot d_X(x,y)$.

1.25 Note: Let X and Y be metric spaces and let $f:X\to Y$ be Lipschitz continuous. Then f is also uniformly continuous. Also, if $x_n\to a$ in X then $f(x_n)\to f(a)$ in Y. Likewise, if (x_n) is Cauchy in X then $\big(f(x_n)\big)$ is Cauchy in Y.

1.26 Theorem: *Let U and V be normed linear spaces and let $F:U\to V$ be a linear map. Then the following are equivalent:*
(1) F is Lipschitz continuous on U.
(2) F is continuous at some point $a\in U$.
(3) F is continuous at 0.
(4) F is bounded.
In this case, $\|F\|$ is a Lipschitz constant for F.

Proof: If F is Lipschitz continuous on U then F is continuous at some point $a\in U$ (indeed F is continuous at every point $a\in U$). If F is continuous at some point $a\in U$ then F is also continuous at 0 because, given $x\in U$, if we let $u=x+a$ then we have $\|x\|=\|u-a\|$ and $\|Fx\|=\|Fu-Fa\|$. If F is continuous at 0 then F is bounded because if we choose $\delta>0$ such that for all $x\in U$ with $\|x\|\leq\delta$ we have $\|Fx\|\leq 1$, then for all $x\in U$ with $\|x\|=1$ we have $\|\delta x\|=\delta$ so that $\|Fx\|=\frac{1}{\delta}\|F(\delta x)\|\leq\frac{1}{\delta}$, and it follows that $\|F\|\leq\frac{1}{\delta}$. Finally note that if F is bounded then for all $x,y\in U$ we have

$$d(Fx,Fy)=\|Fx-Fy\|=\|F(x-y)\|\leq\|F\|\,\|x-y\|=\|F\|\,d(x,y)$$

so that F is Lipschitz continuous with Lipschitz constant $\|F\|$.

Dual Spaces

1.27 Definition: The (linear) **dual space** of a vector space U over a field $\mathbb{F}$ is the vector space

$$U^{\#} = \text{Hom}(U,\mathbb{F}) = \big\{f : U \to \mathbb{F}\,\big|\, f \text{ is linear}\big\}.$$

The (continuous) **dual space** of a normed linear space U over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is the normed linear space

$$U^* = \mathcal{B}(U,\mathbb{F}) = \big\{f : U \to \mathbb{F}\,\big|\, f \text{ is linear with } \|f\| < \infty\big\}$$

using the operator norm. Note U^* is a Banach space by Theorem 1.23.

1.28 Theorem: *(The Riesz Representation Theorem for the ℓ_p Spaces) Let $p, q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

(1) The map $F : \ell_q \to {\ell_p}^*$ given by $F(b)(a) = \sum\limits_{k=1}^{\infty} a_k b_k$ is well-defined, linear, injective and norm-preserving.

(2) When $p \neq \infty$, the above map F is also surjective, so we have ${\ell_p}^* \cong \ell_q$.

Proof: For $b \in \ell_q$, write $F_b = F(b)$. For $a \in \ell_p$, writing $|a| = \big(|a_1|, |a_2|, \cdots\big)$ and similarly for $|b|$, Hölder's Inequality gives

$$\big|F_b(a)\big| = \Big|\sum_{k=1}^{\infty} a_k b_k\Big| \leq \sum_{k=1}^{\infty} |a_k b_k| = \big\||a|\,|b|\big\|_1 \leq \|a\|_p \|b\|_q$$

so that F_b is a well-defined bounded linear map $F_b : \ell_p \to \mathbb{F}$ with $\|F_b\| \leq \|b\|_q$, and hence F is a well-defined linear map $F : \ell_q \to {\ell_p}^*$. We claim that F preserves norm.

Case 1: suppose $p = 1$ (and $q = \infty$). Let $b \in \ell_\infty$. Let $e_n = (0, \cdots, 0, 1, 0, \cdots)$ be the n^{th} standard basis vector in $\mathbb{F}^\infty$. Then $\|e_n\|_1 = 1$ and $\big|F_b(e_n)\big| = |b_n|$ hence $\|F_n\| \geq |b_n|$. Since $\|F_b\| \geq b_n$ for all $n \in \mathbf{Z}^+$, we have $\|F_b\| \geq \sup\big\{|b_n|\,\big|\, n \in \mathbf{Z}^+\big\} = \|b\|_\infty$. We already showed above that $\|F_b\| \leq \|b\|_\infty$ so we have $\|F_b\| = \|b\|_\infty$, and so F is norm-preserving.

Case 2: suppose $1 < p < \infty$. Let $b \in \ell_q$. If $b = 0$ then $F_b = 0$. Suppose that $b \neq 0$. Let $m \in \mathbf{Z}^+$ be large enough so $b_k \neq 0$ for some $k \leq m$. Let $a = (a_1, a_2, \cdots, a_m, 0, 0, \cdots) \in \ell_p$ where $a_k = \frac{|b_k|^q}{b_k}$ when $k \leq m$ and $b_k \neq 0$ and $a_k = 0$ otherwise. Since $\big|F_b(a)\big| \leq \|F_b\|\,\|a\|_p$, and $F_b(a) = \sum\limits_{k=1}^{m} |b_k|^q$, and $\|a\|_p = \Big(\sum\limits_{k=1}^{m} |b_k|^{p(q-1)}\Big)^{1/p} = \Big(\sum\limits_{k=1}^{m} |b_k|^q\Big)^{1/p}$, we have

$$\sum_{k=1}^{m} |b_k|^q = \big|F_b(a)\big| \leq \|F_b\|\,\|a\|_p = \|F_b\|\Big(\sum_{k=1}^{m} |b_k|^q\Big)^{1/p}$$

and hence

$$\|F_b\| \geq \Big(\sum_{k=1}^{m} |b_k|^q\Big)^{1-\frac{1}{p}} = \Big(\sum_{k=1}^{m} |b_k|^q\Big)^{1/q} \longrightarrow \|b\|_q \text{ as } m \to \infty.$$

It follows that $\|F_b\| \geq \|b\|_q$ hence $\|F_b\| = \|b\|_q$, and so F preserves norm.

Case 3: suppose $p = \infty$ (and $q = 1$). Let $b \in \ell_1$. For each $k \in \mathbf{Z}^+$, let $a_k = \frac{|b_k|}{b_k}$ if $b_k \neq 0$ and let $a_k = 1$ if $b_k = 0$, and let $a = (a_1, a_2, \cdots)$. Then we have $\|a\|_\infty = 1$ and $F_b(a) = \sum\limits_{k=1}^{\infty} |b_k| = \|b\|_1$, and so $\|F_b\| \geq \|b\|_1$. Thus $\|F_b\| = \|b\|_1$ so F preserves norm.

Since every norm-preserving map is injective, this proves Part 1. To prove Part 2, let $1 \leq p < \infty$ and let $f \in {\ell_p}^*$. Let $b = \big(f(e_1), f(e_2), f(e_3), \cdots\big) \in \mathbb{F}^\omega$, where e_k is the k^{th} standard basis vector in $\mathbb{F}^\infty$. We claim that $b \in \ell_q$ and that $F_b = f$, so that F is surjective, as required.

Case 1: suppose $p = 1$ (and $q = \infty$). Let $f \in {\ell_1}^*$ and let $b = \big(f(e_1), f(e_2), \cdots\big) \in \mathbb{F}^\omega$. For each $k \in \mathbf{Z}^+$ we have $\|e_k\|_1 = 1$ hence $|b_k| = \big|f(e_k)\big| \leq \|f\|$. Since $|b_k| \leq \|f\|$ for all $k \in \mathbf{Z}^+$, we have $\|b\|_\infty \leq \|f\| < \infty$ and so $b \in \ell_\infty$, as required. To show that $F_b = f$, let $a \in \ell_p$. For each $m \in \mathbf{Z}^+$, let $x_m = \sum\limits_{k=1}^{m} a_k e_k = (a_1, \cdots, a_m, 0, 0, \cdots) \in \ell_p$. Note that $x_m \to a$ in ℓ_p. Since f is continuous and linear we have

$$f(a) = \lim_{m\to\infty} f(x_m) = \lim_{m\to\infty} \sum_{k=1}^{m} a_k f(e_k) = \lim_{m\to\infty} \sum_{k=1}^{m} a_k b_k = \sum_{k=1}^{\infty} a_k b_k = F_b(a).$$

Case 2: suppose $1 < p < \infty$. Let $f \in {\ell_p}^*$ and let $b = \big(f(e_1), f(e_2), \cdots\big) \in \mathbb{F}^\omega$. If $b = 0$ then $b \in \ell_q$. Suppose $b \neq 0$. As in our proof in Case 2 of Part 1, choose $m \in \mathbf{Z}^+$ large enough that $b_k \neq 0$ for some $k \leq m$ and let $a = (a_1, \cdots, a_m, 0, 0, \cdots)$ with $a_k = \frac{|b_k|^q}{b_k}$ when $k \leq m$, $b_k \neq 0$. As above, we have $\sum\limits_{k=1}^{m} |b_k|^q = \big|f(a)\big| \leq \|f\|\,\|a\|_p = \Big(\sum\limits_{k=1}^{m} |b_k|^q\Big)^{1/p}$ and hence $\|f\| \geq \Big(\sum\limits_{k=1}^{m} |b_k|^q\Big)^{1/q}$, and it follows, by taking the limit as $m \to \infty$, that $\|b\|_q \leq \|f\| < \infty$ so that $b \in \ell_q$, as required. We can show that $F_b = f$ as we did in Case 1. Given $a \in \ell_p$, we let $x_m = \sum\limits_{k=1}^{m} a_k e_k \in \ell_p$, we note that $x_m \to a$ in ℓ_p, and use the fact that f is continuous and linear to get $f(a) = F_b(a)$.

1.29 Remark: In the case that $p = \infty$ and $q = 1$, the above proof breaks down at the last step, because when $a \in \ell_\infty$ and $x_m = (a_1, \cdots, a_m, 0, 0, \cdots)$, we do not have $x_m \to a$ in ℓ_∞. In fact, the map F is not surjective in this case (as we shall see later, using the Hahn-Banach Theorem).

1.30 Remark: We often identify ${\ell_p}^*$ with its image under the above map F, so we identify ℓ_q with ${\ell_p}^*$ when $1 \leq p < \infty$, and we identify ℓ_1 with a (proper) subspace of ${\ell_\infty}^*$.

1.31 Theorem: *(The Riesz Representation Theorem for the L_p Spaces) Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, and let $A \subseteq \mathbb{R}$ be measurable with $\lambda(A) > 0$.*

(1) The map $F : L_q(A) \to L_p(A)^*$ given by $F(g)(f) = \int_A fg$ is a well-defined injective norm-preserving map.

(2) When $1 \leq p < \infty$, the above map F is also surjective, so we have $L_p(A)^* \cong L_q(A)$.

Proof: At the moment we can only prove Part 1, because we have not developed sufficient machinery to prove Part 2. Part 2 is often proven in PMATH 451 using the Radon-Nikodym Theorem. We might give a proof later in the course using an alternate method.

To prove Part 1, let $g \in L_q(A)$ and write $F_g = F(g)$. For $f \in L_p(A)$, by Hölder's Inequality we have

$$\big|F_g(f)\big| = \left|\int_A fg\right| \leq \int_A |fg| = \big\|\,|f|\,|g|\,\big\|_1 \leq \|f\|_p \|g\|_q \leq \|g\|_q$$

so that F_g is a well-defined bounded linear map $F_g : \ell_p \to \mathbb{F}$ with $\|F_g\| \leq \|g\|_q$. Thus F is a well-defined linear map $F : L_q(A) \to L_p(A)^*$. It remains to show that $\|F_g\| \geq \|q\|_q$ for all $g \in L_q(A)$, so that F preserves norm.

Case 1: suppose that $p = 1$ (and $q = \infty$). Let $g \in L_q(A)$. Let $\epsilon > 0$ and let $C = \{x \in A \,|\, |g(x)| > \|g\|_\infty - \epsilon\}$. Note that, by the definition of $\|g\|_\infty$, we have $\lambda(C) > 0$. Choose $B \subseteq C$ so that $0 < \lambda(B) < \infty$. Define $s : A \to \mathbb{F}$ by $s(x) = \frac{g(x)}{|g(x)|}$ if $g(x) \neq 0$ and $s(x) = 1$ if $g(x) = 0$ (so that we have $sg = |g|$), and let $f = \dfrac{s}{\lambda(B)}\,\chi_B$ where χ_B is the characteristic function of B (given by $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ if $x \neq B$). Then

$$\|f\|_1 = \int_A \frac{|s|}{\lambda(B)}\,\chi_B = \frac{1}{\lambda(B)} \int_A \chi_B = 1$$

and

$$\big|F_g(f)\big| = \left|\int_A fg\right| = \int_A \frac{sg}{\lambda(B)}\,\chi_B = \frac{1}{\lambda(B)} \int_B |g| \geq \frac{1}{\lambda(B)} \int_B \big(\|g\|_\infty - \epsilon\big) = \|g\|_\infty - \epsilon.$$

It follows that $\|F_g\| \geq \|g\|_\infty - \epsilon$ for every $\epsilon > 0$, and so $\|F_g\| \geq \|g\|_\infty$, as required.

Case 2. suppose that $1 < p < \infty$. Let $g \in L_q(A)$. If $g = 0$ then $\|F_g\| \geq \|g\|_q$. Suppose that $\|g\|_q \neq 0$. Define $s : A \to \mathbb{F}$ as above, by $s(x) = \frac{g(x)}{|g(x)|}$ if $g(x) \neq 0$ and $s(x) = 1$ if $g(x) = 0$, and let $f = \dfrac{s}{\|g\|_q^{q/p}}|g|^{q/p}$. Then

$$\|f\|_p^p = \int_A |f|^p = \int_A \frac{1}{\|g\|_q^q}\,|g|^q = 1$$

and, since $\frac{q}{p} + 1 = q\big(\frac{1}{p} + \frac{1}{q}\big) = q$ hence also $q - \frac{q}{p} = 1$, we have

$$\begin{aligned}\big|F_g(f)\big| &= \left|\int_A fg\right| = \left|\int_A \frac{sg}{\|g\|_q^{q/p}}\,|g|^{q/p}\right| = \int_A \frac{|g|}{\|g\|_q^{q/p}}\,|g|^{q/p} = \frac{1}{\|g\|_q^{q/p}} \int_A |g|^{q/p+1} \\ &= \frac{1}{\|g\|_q^{q/p}} \int_A |g|^q = \frac{1}{\|g\|_q^{q/p}} \|g\|_q^q = \|g\|_q^{q-q/p} = \|g\|_q\end{aligned}$$

so we have $\|F_g\| \geq \|g\|_q$, as required.

Case 3. suppose that $p = \infty$ (and $q = 1$). Let $g \in L_1(A)$. Define $s : A \to \{\pm 1\}$ as above, by $s(x) = \frac{g(x)}{|g(x)|}$ if $g(x) \neq 0$ and $s(x) = 1$ if $g(x) = 0$. Then $\|s\|_\infty = 1$ and $\big|F_g(s)\big| = \int_A |g| = \|g\|_1$ so that $\|F_g\| \geq \|g\|_1$, as required.

Uniform Boundedness

1.32 Definition: Let X be a metric space and let $A \subseteq X$. Recall that A is **dense** (in X) when for every nonempty open ball $B \subseteq X$ we have $B \cap A \neq \emptyset$, equivalently when $\overline{A} = X$. We say A is **nowhere dense** (in X) when for every nonempty open ball $B \subseteq \mathbb{R}$ there exists a nonempty open ball $C \subseteq B$ with $C \cap A = \emptyset$, or equivalently when $\overline{A}^{\,0} = \emptyset$.

When $A \subseteq B \subseteq X$, note that if A is dense in X then so is B and, on the other hand, if B is nowhere dense in X then so is A. When $A, B \subseteq X$ with $B = A^c = X \setminus A$, note that A is nowhere dense $\iff \overline{A}^{\,0} = \emptyset \iff \overline{B^0} = X \iff$ the interior of B is dense.

1.33 Definition: Let $A \subseteq X$. We say that A is **first category** (or that A is **meagre**) when A is equal to a countable union of nowhere dense sets. We say that A is **second category** when it is not first category. We say that A **residual** when A^c is first category.

Note that every countable set in $\mathbb{R}$ is first category since if $A = \{a_1, a_2, a_3, \cdots\}$ then we have $A = \bigcup_{k=1}^{\infty} \{a_k\}$. In particular $\mathbb{Q}$ is first category and $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$ is residual. Also note that if $A \subseteq X$ is first category then so is every subset of A. and that if $A_1, A_2, A_3, \cdots \subseteq X$ are are all first category then so is $\bigcup_{k=1}^{\infty} A_k$.

1.34 Theorem: *(The Baire Category Theorem) Let X be a complete metric space.*
(1) Every first category set in X has an empty interior.
(2) Every residual set in X is dense.
(3) Every countable union of closed sets with empty interiors in X has an empty interior.
(4) Every countable intersection of dense open sets in X is dense.

Proof: Parts (1) and (2) are equivalent by taking complements, and Parts (3) and (4) are special cases of Parts (1) and (2), so it suffices to prove Part (1). We sketch a proof.

Let $A \subseteq X$ be first category, say $A = \bigcup_{n=1}^{\infty} C_n$ where each C_n is nowhere dense. Suppose, for a contradiction, that A has nonempty interior, and choose an open ball $B_0 = B(a_0, r_0)$ with $0 < r_0 < 1$ such that $\overline{B}_0 \subseteq A$. Since each C_n is nowhere dense, we can chose a nested sequence of open balls $B_n = B(a_n, r_n)$ with $0 < r_n < \frac{1}{2^n}$ such that $\overline{B}_n \subseteq B_{n-1}$ and $\overline{B}_n \cap C_n = \emptyset$. Because $r_n \to 0$, it folows that the sequence $\{a_n\}$ is Cauchy. Because X is complete, it follows that $\{a_n\}$ converges in X, say $a = \lim_{n\to\infty} a_n$. Note that $a \in \overline{B}_n$ for all n since $a_k \in \overline{B}_n$ for all $k \geq n$. Since $a \in \overline{B}_0$ and $\overline{B}_0 \subseteq A$ we have $a \in A$. But since $a \in \overline{B}_n$for al $n \geq 1$, and $\overline{B}_n \cap C_n = \emptyset$, we have $a \notin C_n$ for all $n \geq 1$ hence $a \notin \bigcup_{n=1}^{\infty} C_n$, that is $a \notin A$.

1.35 Example: Recall that $\mathbb{Q}$ is first category and $\mathbb{Q}^c$ is residual. The Baire Category Theorem shows that $\mathbb{Q}^c$ cannot be first category because if $\mathbb{Q}$ and $\mathbb{Q}^c$ were both first category then $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ would also be first category, but this is not possible since $\mathbb{R}$ does not have empty interior.

1.36 Notation: Let X be a set. For any set $\mathcal{C}$ of subsets of X we write

$$\mathcal{C}_\sigma = \left\{ \bigcup_{k=1}^{\infty} A_k \,\middle|\, \text{each } A_k \in \mathcal{C} \right\} \text{ and } \mathcal{C}_\delta = \left\{ \bigcap_{k=1}^{\infty} A_k \,\middle|\, \text{each } A_k \in \mathcal{C} \right\}.$$

Note that $\mathcal{C}_{\sigma\sigma} = \mathcal{C}_\sigma$ and $\mathcal{C}_{\delta\delta} = \mathcal{C}_\delta$.

1.37 Definition: Let X be a set. A **σ-algebra** in X is a set $\mathcal{C}$ of subsets of X such that
(1) $\emptyset \in \mathcal{C}$,
(2) if $A \in \mathcal{C}$ then $A^c = X \setminus A \in \mathcal{C}$, and
(3) if $A_1, A_2, A_3, \dots \in \mathcal{C}$ then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{C}$.

Note that when $\mathcal{C}$ is a σ-algebra in X we have $\mathcal{C}_\sigma = \mathcal{C}$ and $\mathcal{C}_\delta = \mathcal{C}$.

1.38 Notation: In a metric space (or topological space) X, we let $\mathcal{G}$ denote the set of all open sets in X and we let $\mathcal{F}$ denote the set of all closed subsets of X. Note that $\mathcal{G}_\sigma = \mathcal{G}$ and $\mathcal{F}_\delta = \mathcal{F}$.

1.39 Exercise: Using the Baire Category Theorem, show that in $\mathbb{R}$ we have $\mathcal{F} \subseteq \mathcal{G}_\delta$ (equivalently $\mathcal{G} \subseteq \mathcal{F}_\sigma$), $\mathcal{F}_\sigma \neq \mathcal{G}_\delta$, and $\mathcal{G}_\delta \cup \mathcal{F}_\sigma \subsetneqq \mathcal{G}_{\delta\sigma} \cap \mathcal{F}_{\sigma\delta}$.

1.40 Theorem: *(The Banach-Steinhaus Theorem, or the Uniform Boundedness Principle) Let X be a Banach space and let Y be a normed linear space. Let S be a set of bounded linear maps $L : X \to Y$. Suppose that for every $x \in X$ there exists $m_x \geq 0$ such that $||Lx|| \leq m_x$ for all $L \in S$. Then there exists $m \geq 0$ such that $||L|| \leq m$ for all $L \in S$.*

Proof: For each $n \in \mathbf{Z}^+$, let $A_n = \left\{x \in X \,\middle|\, ||Lx|| \leq n \text{ for all } L \in S\right\}$. Note that A_n is closed because the sets $\left\{x \in X \,\middle|\, ||Lx|| \leq n\right\}$ are closed for each $L \in S$, and A_n is the intersection of these sets. By the hypothesis of the theorem, we have $X = \bigcup_{n=1}^{\infty} A_n$. By the Baire Category Theorem (since X is complete), the sets A_n cannot all be nowhere dense. Choose $n \in \mathbf{Z}^+$ so that A_n is not nowhere dense. Chose $a \in A_n$ and $r > 0$ so that $\overline{B}(a, r) \subseteq A_n$. For all $x \in X$, if $x \in B(a, r)$ then $x \in A_n$ so we have $||L(x)|| \leq n$ for all $L \in S$. If $||x|| < r$ then $x + a \in B(a, r)$ and $a \in B(a, r)$ and so

$$||L(x)|| = ||L(x + a) - L(a)|| \leq ||L(x + a)|| + ||L(a)|| \leq 2n \text{ for all } L \in S.$$

For all $L \in S$ and $x \in X$, if $||x|| \leq 1$ then $||rx|| \leq r$ and so $||L(x)|| = \frac{1}{r}\,||L(rx)|| \leq \frac{2n}{r}$. Thus we have $||L|| \leq \frac{2n}{r}$ for all $L \in S$.

1.41 Theorem: *(Condensation of Singularities) Let X be a Banach space and let Y be a normed linear space. For each $m, n \in \mathbf{Z}^+$, let $L_{m,n} : X \to Y$ be a bounded linear map. Suppose that for each $m \in \mathbf{Z}^+$ there exists $x_m \in X$ such that $\limsup_{n \to \infty} ||L_{mn}(x_m)|| = \infty$. Then the set $E = \left\{x \in X \,\middle|\, \limsup_{n \to \infty} ||L_{mn}(x)|| = \infty \text{ for all } m \in \mathbf{Z}^+\right\}$ is a dense $\mathcal{G}_\delta$ set.*

Proof: Fix $m \in \mathbf{Z}^+$. For each $\ell \in \mathbf{Z}^+$, let $A_\ell = \left\{x \in X \,\middle|\, ||L_{n,m}(x)|| \leq \ell \text{ for all } n \in \mathbf{Z}^+\right\}$ and note that each set A_ℓ is closed. As in the proof of the Uniform Boundedness Principle, if one of the sets A_ℓ was not nowhere dense then we could choose $m \geq 0$ such that $||L_{m,n}|| \leq m$ for all $n \in \mathbf{Z}^+$. But then for all $x \in X$ we would have $||L_{m,n}(x)|| \leq m||x||$ for all n so that $\limsup_{n \to \infty} ||L_{m,n}(x)|| \leq m||x||$, contradicting the hypothesis of the theorem. Thus all of the sets A_ℓ must be nowhere dense. Let $B_m = \bigcup_{\ell=1}^{\infty} A_\ell = \left\{x \in X \,\middle|\, \limsup_{n \to \infty} ||L_{m,n}(x)|| < \infty\right\}$ and let $C = \bigcup_{m=1}^{\infty} B_m = \left\{x \in X \,\middle|\, \limsup_{n \to \infty} ||L_{m,n}(x)|| < \infty \text{ for some } m \in \mathbf{Z}^+\right\}$, and note that $E = X \setminus C$. Then C is a countable union of closed nowhere dense sets, so E is a countable intersection of open dense sets. By the Baire Category Theorem, E is dense.