Chapter 5. Fourier Series

5.1 Remark: We shall begin with an informal discussion of Fourier series and how they can be used in physics and engineering.

5.2 Definition: A real **trigonometric series** is a series of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_n, b_n \in \mathbb{R}$ and $x \in \mathbb{R}$. If the series converges, we say it is the real **Fourier series** of its sum

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$

which is a periodic function of the real variable x with period 2π, and the numbers a_n, b_n are called the **Fourier coefficients** of $f(x)$. If we are justified in integrating term by term then, using the formulas

$$\int_{-\pi}^{\pi} 1 = 2\pi \ , \ \int_{-\pi}^{\pi} \cos nx \ dx = 0 = \int_{-\pi}^{\pi} \sin nx \ dx \ , \ \int_{-\pi}^{\pi} \cos^2 nx \ dx = \pi = \int_{-\pi}^{\pi} \sin^2 nx \ dx$$

$$\int_{-\pi}^{\pi} \cos nx \ \cos mx \ dt = 0 = \int_{-\pi}^{\pi} \sin nx \ \sin mx \ dt \ , \text{ and } \int_{-\pi}^{\pi} \cos nx \ \sin mx \ dt = 0$$

where $n, n \in \mathbb{Z}^+$ with $n \neq m$, we find that the Fourier coefficients are given by

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \ , \ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \ dx \ , \text{ and } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \ dx.$$

5.3 Remark: For the moment, we shall blithely assume that, given a 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$, the Fourier series with coefficients a_n and b_n given by the above formulas converges to the given function $f(x)$.

5.4 Example: Find the Fourier coefficients of the 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} \frac{\pi}{2} + x \text{ for } -\pi \leq x \leq 0, \\ \frac{\pi}{2} - x \text{ for } \ 0 \leq x \leq \pi. \end{cases}$$

Solution: Since $f(x)$ is even, we have $a_n = 0$ for all $n \in \mathbb{Z}^+$, and we have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\pi} \tfrac{\pi}{2} - x \ dx = \tfrac{2}{\pi} \Big[\tfrac{\pi}{2} x - \tfrac{1}{2} x^2 \Big]_0^{\pi} = 0$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \ dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \ dx = \frac{2}{\pi} \int_0^{\pi} \left(\tfrac{\pi}{2} - x \right) \cos nx \ dx \\ &= \int_0^{\pi} \cos nx \ dx - \frac{2}{\pi} \int_0^{\pi} x \ \cos nx \ dx = \Big[\tfrac{1}{n} \sin nx \Big]_0^{\pi} - \tfrac{2}{\pi} \Big[\tfrac{1}{n} x \ \sin nx + \tfrac{1}{n^2} \cos nx \Big]_0^{\pi} \\ &= 0 - \tfrac{2}{\pi} \left(\tfrac{1}{n^2} (-1)^n - 1 \right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{4}{\pi\, n^2} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Thus, assuming that the Fourier series of $f(x)$ converges to $f(x)$, we have

$$f(x) = \tfrac{4}{\pi} \left(\tfrac{1}{1^2} \cos x + \tfrac{1}{3^2} \cos 3x + \tfrac{1}{5^2} \cos 5x + \cdots \right)$$

5.5 Remark: Assuming convergence, putting $x = 0$ into the above function $f(x)$ gives $\frac{\pi}{2} = \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$ so we obtain the formula

$$\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

5.6 Example: (Forced Damped Oscillations) Suppose an object of mass m is attached to a spring of spring-constant k and vibrates in a fluid of damping-constant c and let $x = x(t)$ be the displacement of the object from its equilibrium position at time t. Suppose, in addition, that the object is acted on by an external force $f(t)$. The total force $F(t)$ acting on the object consists of the force exerted by the spring, which is equal to $-kx(t)$, the resistive force exerted by the fluid, which is equal to $-cx'(t)$, and the external driving force, which is equal to $f(t)$. By Newton's Second Law of motion we have $F(t) = mx''(t)$ and so $x(t)$ satisfies the differential equation (the DE)

$$mx''(t) + cx'(t) + kx(t) = f(t).$$

5.7 Example: Use Fourier series to solve the above DE with $m = 1$, $c = 2$ and $k = 5$, where $f(t)$ is the function from Example 5.4,

Solution: We need to solve the DE

$$x''(t) + 2x'(t) + 5x(t) = f(t).$$

To solve the associated homogeneous DE $x'' + 2x' + 5x = 0$, we look for a solution of the form $x = x(t) = e^{rt}$. Putting $x = e^{rt}$, $x' = re^{rt}$ and $x'' = r^2e^{rt}$ into the homogeneous DE gives $(r^2+2r+5)e^{rt} = 0$ hence $r = -1 \pm 2i$. This gives us the two complex-valued solutions $x(t) = e^{(-1\pm 2i)t} = e^{-t}(\cos 2t \pm i \sin 2t)$. By taking suitable linear combinations of these two complex-valued solutions we obtain the two real-valued solutions $x_1(t) = e^{-t}\cos 2t$ and $x_2(t) = e^{-t}\sin 2t$. The general solution to the DE $x'' + 2x' + 5x = 0$ is given by

$$x(t) = Ae^{-t}\cos 2t + Be^{-t}\sin 2t \quad \text{, where } A, B \in \mathbb{R}.$$

For each $n \in \mathbb{Z}^+$, to find a particular solution to the DE $x'' + 2x' + 5x = \cos nt$, we look for a solution of the form $x = x(t) = A_n \cos nt + B_n \sin nt$. Putting $x = A_n \cos nt + B_n \sin nt$, $x' = -nA_n \sin nx + nB_n \cos nt$ and $x'' = -n^2A_n \cos nt - n^2B_n \sin nt$ into $x'' + 2x' + 5 = \cos nt$ gives $\big(-n^2A_n + 2nB_n + 5A_n\big)\cos nt + \big(-n^2B_n - 2nA_n + 5B_n\big)\sin nt = \cos nt$ for all $t \in \mathbb{R}$ and so we must have $(5-n^2)A_n + 2nB_n = 1$ and $(5-n^2)B_n - 2nA_n = 0$. We solve these two equations to get $A_n = \frac{5-n^2}{n^4-6n^2+25}$ and $B_n = \frac{2n}{n^4-6n^2+25}$ and so one solution to the DE $x'' + 2x' + 5 = \cos nt$ is given by

$$x(t) = A_n \cos nt + B_n \sin nx \text{ , where } A_n = \frac{5-n^2}{n^4-6n^2+25} \text{ and } B_n = \frac{2n}{n^4-6n^2+25}.$$

Since $f(t) = \sum_{n \text{ odd}} \frac{4}{\pi n^2} \cos nt$, one particular solution, called the **steady state solution**, to the original DE $x'' + 2x' + 5x = f(t)$ is given by

$$x(t) = \sum_{n \text{ odd}} \frac{4}{\pi n^2}\big(A_n \cos nt + B_n \sin nt\big)$$

and the general solution is

$$x(t) = Ae^{-t}\cos 2t + Be^{-t}\sin 2t + \sum_{n \text{ odd}} \frac{4}{\pi n^2}\big(A_n \cos nt + B_n \sin nt\big) \quad \text{, where } A, B \in \mathbb{R}.$$

5.8 Example: (The One-Dimensional Wave Equation) An elastic string is stretched to length π and is fixed at its two endpoints along the x-axis at $x = 0$ and $x = \pi$. The string is displaced so that it follows the curve $u = f(x)$ with $f(0) = 0$ and $f(\pi) = 0$, then at time $t = 0$ the string is released and allowed to vibrate. The problem is to determine the strings shape $u = u(x,t)$ at all points $0 \le x \le \pi$ and all times $t \ge 0$.

To formulate a differential equation (or DE) which models the situation, we consider a segment of string, at time t, between the points $p_1 = (x_1, u(x_1,t))$ and $p_2 = (x_2, u(x_2,t))$ where the difference $dx = x_2 - x_1$ is small. The slope of the curve $u = g(x) = u(x,t)$ at p_1 is $\frac{\partial u}{\partial x}(x_1,t)$ and the angle θ_1 from the horizontal is given by $\tan\theta_1 = \frac{\partial u}{\partial x}(x_1,t)$. Similarly, the angle θ_2 at p_2 is given by $\tan\theta_2 = \frac{\partial u}{\partial x}(x_2,t)$, and we have

$$\tan\theta_2 - \tan\theta_1 = \frac{\partial u}{\partial x_1}(x_1,t) - \frac{\partial u}{\partial x}(x_2,t) = \frac{\partial^2 u}{\partial x^2}\,dx.$$

Let T_1 be the magnitude of the force exerted on p_1 by the portion of the string which lies to the left of p_1, and let T_2 be the magnitude of the force exerted on p_2 by the portion of the string which lies to the right of p_2. Assuming that the segment of string moves only vertically (so the total horizontal component of the force is zero) we have $T_1\cos\theta_1 = T_2\cos\theta_2$. Let

$$T = T_1\cos\theta_1 = T_2\cos\theta_2$$

and note that T is a constant which we call the **tension** of the string. The total vertical component of the force is $F = T_2\sin\theta_2 - T_1\sin\theta_1$ and by Newton's Second Law of motion, we have

$$T_2\sin\theta_2 - T_2\sin\theta_1 = m\frac{\partial^2 u}{\partial t^2} = \rho\,dx\frac{\partial^2 u}{\partial t^2}$$

where ρ is the linear **density** of the string, that is its mass per unit length. From the equations $\tan\theta_2 - \tan\theta_1 = \frac{\partial^2 u}{\partial x^2}\,dx$, $T_1\cos\theta_1 = T_2\cos\theta_2$ and $T_2\sin\theta_2 - T_1\sin\theta_2 = \rho\,dx\,\frac{\partial^2 u}{\partial t^2}$ we obtain the **one-dimentional wave equation**

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}\ \text{, where } c^2 = \frac{T}{\rho}.$$

5.9 Example: Use Fourier series to solve the one-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ subject to the boundary conditions $u(0,t) = 0$ and $u(0,\pi) = 0$ for all $t \ge 0$ and to the initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$ for all $0 \le x \le \pi$.

Solution: We use a method known as **separation of variables**. We look for a solution to the DE of the form $u(x,t) = y(x)s(t)$ which satisfies the given boundary conditions $0 = u(0,t) = y(0)s(t)$ and $0 = u(\pi,t) = y(\pi)s(t)$. If we had $y(x) = 0$ for all x or $s(t) = 0$ for all t then we would obtain the trivial solution $u(x,t) = 0$ for all x,t, so let us assume this is not the case, so the boundary conditions become $y(0) = y(\pi) = 0$. When $u(x,t) = y(x)s(t)$, the DE becomes $y(x)s''(t) = c^2y''(x)s(t)$ which we can write as $\frac{y''(x)}{y(x)} = \frac{1}{c^2}\,\frac{s''(t)}{s(t)}$. Since the function on the left is a function of x (and is constant in t) and the function on the right is a function of t (and is constant in x), in order for these two functions to be equal for all x,t they must both be constant, say

$$\frac{y''(x)}{y(x)} = k = \frac{1}{c^2}\,\frac{s''(t)}{s(t)}$$

where k is constant.

First we solve the DE $\frac{y''(x)}{y(x)} = k$ subject to the boundary conditions $y(0) = y(\pi) = 0$. If $k = 0$ then the DE becomes $y'' = 0$, which has solution $y = Cx + D$, and the boundary conditions give $C = D = 0$, so we obtain the trivial solution. If $k > 0$, say $k = n^2$ where $n > 0$, then the DE becomes $y'' - n^2 y = 0$, which has solution $y = Ce^{nx} + De^{-nx}$, and the boundary conditions give $C + D = 0$ and $Ce^{n\pi} + De^{-n\pi} = 0$ which imply that $C = D = 0$, so again we obtain the trivial solution. Suppose that $k < 0$, say $k = -n^2$ where $n > 0$. The DE becomes $y'' + n^2 y = 0$ which has solution $y = C\cos nx + D\sin nx$. The boundary condition $y(0) = 0$ gives $C = 0$ so that $y = D\sin nx$, and the boundary condition $y(\pi) = 0$ gives $D = 0$ or $\sin n\pi = 0$. If $D = 0$ we obtain the trivial solution and if $\sin n\pi = 0$ then we must have $n \in \mathbb{Z}$. Thus in order to obtain a nontrivial solution to the DE which satisfies the boundary conditions we must have $k = -n^2$ for some $n \in \mathbb{Z}^+$ and, in this case,

$$y(x) = D_n \sin nx \text{ , where } D_n \in \mathbb{R}.$$

When $k = -n^2$ with $n \in \mathbb{Z}^+$, and $y(x) = D_n \sin nx$, the DE $\frac{1}{c^2}\frac{s''(t)}{s(t)} = k$. becomes $s''(t) + (cn)^2 s(t) = 0$, and the solution is $s(t) = A_n \cos(cn\,t) + B_n \sin(cn\,t)$. Thus, for each $n \in \mathbb{Z}^+$, and for all $A_n, B_n \in \mathbb{R}$, the function

$$u(x,t) = y(x)s(t) = \big(A_n \cos cnt + B_n \sin cnt\big)\sin nx$$

is a solution to the one-dimensional wave equation which satisfies the boundary conditions (we remark that it would be redundant to include the constants D_n as they could be amalgamated with the constants A_n and B_n).

In order to find a solution which satisfies the given initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$, we look for a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} \big(A_n \cos cnt + B_n \sin cnt\big)\sin nx \,.$$

In order to obtain $u(x,0) = f(x)$ we need $\sum_{n=1}^{\infty} A_n \sin nx = f(x)$ and so we choose A_n to be equal to the Fourier coefficients of the odd 2π-periodic function $F(x)$ with $F(x) = f(x)$ for $0 \le x \le \pi$, that is we choose

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx \; dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \; dx \,.$$

Assuming that we can differentiate term-by-term, we have

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty}\big(-cnA_n \sin cnt + cnB_n \cos nt\big)\sin nx.$$

In order to obtain $\frac{\partial u}{\partial t}(x,0) = g(x)$ we need $\sum_{n=1}^{\infty} cnB_n \sin nx = g(x)$ and so we choose B_n to be equal to the Fourier coefficients of the odd 2π-periodic function $G(x)$ with $G(x) = g(x)$ for $0 \le x \le \pi$, that is

$$B_n = \frac{2}{cn\pi}\int_0^{\pi} g(x)\sin nx \; dx \,.$$

5.10 Remark: Let us now begin a more formal presentation of Fourier series in which we consider convergence issues more carefully.

5.11 Definition: A real-valued **trigonometric polynomial** is a function $f:\mathbb{R}\to\mathbb{R}$ of the form

$$f(x)=a_0+\sum_{n=1}^{m}a_n\cos nx+\sum_{n=1}^{m}b_n\sin nx$$

for some $a_n,b_n\in\mathbb{R}$, and we say that $f(x)$ is of degree m when either $a_m\neq 0$ or $b_m\neq 0$. A real-valued **trigonometric series** is a series of the form

$$a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

which is given by its sequence of partial sums $s_m(x)=a_0+\sum\limits_{n=1}^{m}a_n\cos nx+\sum\limits_{n=1}^{m}b_n\sin nx$.

5.12 Remark: A trigonometric series may or may not converge and, indeed, we can consider several different notions of convergence, for example pointwise convergence, uniform convergence, or convergence with respect to a p-norm.

5.13 Definition: Every real-valued trigonometric polynomial is a smooth 2π-periodic function $f:\mathbb{R}\to\mathbb{R}$. Every 2π-periodic function $f:\mathbb{R}\to\mathbb{R}$ determines, and is determined by, a function $f:[-\pi,\pi]\to\mathbb{R}$ with $f(-\pi)=f(\pi)$, or equivalently by a function $f:T\to\mathbb{R}$ where $T=\mathbb{R}/2\pi\mathbb{Z}$, or equivalently by a function $f:S\to\mathbb{R}$ where $S=\left\{e^{it}\big|-\pi\leq t\leq\pi\right\}$ $=\left\{z\in\mathbb{Z}\big||z|=1\right\}$. A function $f:T\to\mathbb{R}$ is continuous (or differentiable, or $\mathcal{C}^k$) if and only if the corresponding 2π-periodic function $f:\mathbb{R}\to\mathbb{R}$ is continuous (or differentiable, or $\mathcal{C}^k$). We say that a function $f:T\to\mathbb{R}$ is **measurable** when the corresponding 2π-periodic function $f:\mathbb{R}\to\mathbb{R}$ is measurable, or equivalently when the corresponding function $f:[-\pi,\pi]\to\mathbb{R}$ with $f(-\pi)=f(\pi)$ is measurable. For a measurable function $f:T\to\mathbb{R}$ and for $1\leq p\leq\infty$ we define the **p-norm** $\|f\|_p$ of the function $f:T\to\mathbb{R}$ to be equal to the p-norm $\|f\|_p$ of the corresponding function $f:[-\pi,\pi]\to\mathbb{R}$. We define $L_p(T,\mathbb{R})$ to be the quotient of the set of measurable functions $f:T\to\mathbb{R}$ with $\|f\|_p<\infty$ under the equivalence relation in which $f\sim g$ when $f(x)=g(x)$ for a.e. $x\in[-\pi,\pi]$. Note that because $\lambda\big([-\pi,\pi]\big)=2\pi<\infty$, for $1\leq p\leq\infty$ we have $L_\infty(T)\subseteq L_p(T)\subseteq L_1(T)$.

5.14 Definition: When $f(x)=a_0+\sum\limits_{n=1}^{m}a_n\cos nx+b_n\sin nx$, where $a_n,b_n\in\mathbb{R}$, we have $f\in\mathcal{C}^\infty(T)$ and we know that the coefficients a_n and b_n are given by the formulas

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx\ ,\ a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\ dx\ ,\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\ dx\,.$$

Note that the above integrals all exist and are finite for any function $f\in L_1(T,\mathbb{R})$. Given a function $f\in L_1(T,\mathbb{R})$, we define the real **Fourier coefficients** of f to be the real numbers $a_n=a_n(f)$ and $b_n=b_n(f)$ given by the above formulas, and we define the real **Fourier series** of f to be the corresponding real trigonometric series. Note that a real Fourier series is a real trigonometric series which arises, in this way, from some function $f\in L_1(T,\mathbb{R})$.

5.15 Definition: A complex-valued **trigonometric polynomial** is a function $f:\mathbb{R}\to\mathbb{C}$ of the form

$$f(x)=\sum_{n=-m}^{m} c_n e^{inx}$$

for some $c_n\in\mathbb{C}$, and we say that $f(x)$ is of degree m when either $c_m\neq 0$ or $c_{-m}\neq 0$. A complex-valued **trigonometric series** is a series of the form

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

which is given by its sequence of partial sums $s_m(x)=\sum\limits_{n=-m}^{m} c_n e^{inx}$.

5.16 Definition: Every complex-valued trigonometric polynomial is a smooth 2π-periodic function $f:\mathbb{R}\to\mathbb{C}$. Every 2π-periodic function $f:\mathbb{R}\to\mathbb{C}$ determines, and is determined by, a function $f:[-\pi,\pi]\to\mathbb{C}$ with $f(-\pi)=f(\pi)$, or equivalently by a function $f:T\to\mathbb{C}$ where $T=\mathbb{R}/2\pi\mathbb{Z}$, or equivalently by a function $f:S\to\mathbb{C}$ where $S=\big\{z\in\mathbb{Z}\big||z|=1\big\}$. For $1\le p\le\infty$, we define $L_p(T)=L_p(T,\mathbb{C})$ in the same way that we defined $L_p(T,\mathbb{R})$. For $f:T\to\mathbb{C}$ given by $f=u+iv$ where $u:T\to\mathbb{R}$ and $v:T\to\mathbb{R}$, f is measurable if and only if u and v are both measurable, and in this case we have $\int_T f=\int_T u+i\int_T v$, $\int_T|f|=\int_T\sqrt{u^2+v^2}$, $\|f\|_p=\|\sqrt{u^2+v^2}\|_p$ and $f\in L_p(T,\mathbb{C})$ if and only if $u\in L_p(T,\mathbb{R})$ and $v\in L_p(T,\mathbb{R})$.

5.17 Definition: When $f(x)=\sum\limits_{n=-m}^{m} c_n e^{inx}$, where $c_n\in\mathbb{C}$, because

$$\int_T e^{ikx}e^{-ilx}dx=\int_{-\pi}^{\pi}\cos(k-l)x\;dx+i\int_{-\pi}^{\pi}\sin(k-l)x\;dx=\begin{cases}2\pi \text{ if } k=l\\ 0 \text{ if } k\neq l,\end{cases}$$

it follows that the coefficients c_n are given by the formula

$$c_n=c_n(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx\,.$$

Note that the above integrals exist and are finite for any function $f\in L_1(T)=L_1(T,\mathbb{C})$. Given a function $f\in L_1(T)$, we define the (complex) **Fourier coefficients** of f to be the complex numbers $c_n=c_n(f)$ given by the above formulas, and we define the (complex) **Fourier series** of f to be the corresponding complex trigonometric series.

5.18 Note: Given $a_n,b_n\in\mathbb{R}$, we have

$$\begin{aligned}a_0+\sum_{n=1}^{m}a_n\cos nx+\sum_{n=1}^{m}b_n\sin nx&=a_0+\sum_{n=1}^{m}a_n\,\tfrac{e^{inx}+e^{-inx}}{2}+\sum_{n=1}^{m}b_n\,\tfrac{e^{inx}-e^{-inx}}{2i}\\&=a_0+\sum_{n=1}^{m}\big(\tfrac{a_0}{2}-i\,\tfrac{b_n}{2}\big)e^{inx}+\sum_{n=1}^{m}\big(\tfrac{a_0}{2}-\tfrac{b_n}{2i}\big)e^{-inx}=\sum_{n=-m}^{m}c_ne^{inx}\end{aligned}$$

where $c_0=a_0$ and $c_n=\frac{1}{2}(a_n-ib_n)$ and $c_{-n}=\overline{c_n}=\frac{1}{2}(a_n+ib_n)$ for $n>0$.

On the other hand, given $f \in L_1(T,\mathbb{R})$, for $n>0$ we have

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_T f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\big(\cos nx - i\sin nx\big)\,dx \\ &= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x)\cos nx - i\int_{-\pi}^{\pi} f(x)\sin nx\ dx\right) = \tfrac{1}{2}(a_n - i\,b_n) \\ c_{-n} &= \frac{1}{2\pi}\int_T f(x)e^{inx}\,dx = \tfrac{1}{2}(a_n + i\,b_n) \end{aligned}$$

It follows that when $f \in L_1(T,\mathbb{R})$, the m^{th} partial sum of the real Fourier series for f is exactly equal to the m^{th} partial sum for the complex Fourier series for f.

5.19 Definition: For $f \in L_1(T) = L_1(T,\mathbb{C})$ we denote the m^{th} partial sum of the Fourier series of f by $s_m(f)$, so we have

$$s_m(f)(x) = \sum_{n=-m}^{m} c_n e^{inx} \quad \text{, where } c_n = c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt.$$

5.20 Exercise: Show that if $f \in L_p(T)$ with $1 \le p \le \infty$, and $s_m(x) = \sum\limits_{n=-m}^{m} d_n\, e^{inx}$ with $s_m \to f$ in $L_p(T)$, then $d_n = c_n(f)$.

5.21 Theorem: *(The Stone-Weierstrass Theorem) Let X be a compact metric space and let $C(X) = C(X,F)$ be the set of continuous functions $f : X \to F$ where $F = \mathbb{R}$ or $\mathbb{C}$. Let A be an algebra in $C(X)$ which contains the constant functions and which separates points in X and is closed under conjugation. Then A is uniformly dense in $C(X)$, which means that for all $f \in C(X)$ and for all $\epsilon > 0$ there exists $g \in A$ such that $\|g - f\|_\infty < \epsilon$.*

Proof: We omit the proof.

5.22 Corollary: *The set of polynomials $\mathbb{R}[x]$ is uniformly dense in $C\big([a,b]\big)$.*

5.23 Corollary: *The set of functions of the form*

$$u(x,y) = \sum_{k=1}^{n} f_k(x)g_k(y) \quad \text{, where } f_k \in C([a,b]) \text{ and } g_k \in C\big([c,d]\big)$$

is uniformly dense in $C\big([a,b]\times[c,d]\big)$.

5.24 Corollary: *The set of real trigonometric polynomials is uniformly dense in $C\big(T,\mathbb{R}\big)$, and the set of complex trigonometric polynomials is uniformly dense in $C(T) = C(T,\mathbb{C})$.*

5.25 Corollary: *(The Riemann-Lebesgue Lemma) Let $f \in L_1(T)$. Then $\lim\limits_{n\to\pm\infty} c_n(f) = 0$.*

Proof: Let $\epsilon > 0$. Since the space of trigonometric polynomials is dense in $C(T)$ using the ∞-norm, hence also dense in $C(T)$ using the 1-norm, and $C(T)$ is dense in $L_1(T)$ using the 1-norm, we can choose a trigonometric polynomial $p(x) = \sum\limits_{n=-m}^{m} a_n e^{inx}$ with $\|p - f\|_1 < \frac{\epsilon}{2\pi}$. Then for $|n| > m$ we have $c_n(p) = a_n = 0$ and so

$$\begin{aligned} |c_n(f)| = \big|c_n(f) - c_n(p)\big| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \big(f(x) - p(x)\big)\, e^{-inx}\,dx\right| \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|f(x) - p(x)\big|\,dx = \tfrac{1}{2\pi}\|f - p\|_1 < \epsilon. \end{aligned}$$

5.26 Note: Since real trigonometric polynomials are dense in $C(T,\mathbb{R})$, hence also in $L_2(T,\mathbb{R})$, it follows that the orthonormal set

$$\left\{\tfrac{1}{\sqrt{2\pi}}\,,\ \tfrac{1}{\sqrt{\pi}}\cos nx\,,\ \tfrac{1}{\sqrt{\pi}}\sin nx\,\middle|\,n\in\mathbb{Z}^+\right\}$$

is a Hilbert basis for the Hilbert space $L_2(T,\mathbb{R})$. For $f\in L_2(T,\mathbb{R})$ we have

$$a_0(f)=\tfrac{1}{2\pi}\langle f,1\rangle\ ,\ a_n(f)=\tfrac{1}{2\pi}\langle f,\cos nx\rangle\ ,\ b_n=\tfrac{1}{2\pi}\langle f,\sin nx\rangle.$$

Similarly, since complex trigonometric polynomials are dense in $L_2(T)=L_2(T,\mathbb{C})$, it follows that the orthonormal set

$$\left\{\tfrac{1}{\sqrt{2\pi}}e^{inx}\,\middle|\,n\in\mathbb{Z}\right\}$$

is a Hilbert basis for the Hilbert space $L_2(T,\mathbb{C})$. For $f\in L_2(T,\mathbb{C})$ we have

$$c_n(f)=\tfrac{1}{2\pi}\langle f,e^{inx}\rangle.$$

The following theorem is an immediate consequence of our earlier study of Hilbert spaces.

5.27 Theorem: *In the Hilbert space $L_2(T)=L_2(T,\mathbb{C})$, we have the following.*

(1) (Best Approximation) Given $f\in L_2(T)$, $s_m(f)$ is the unique trigonometric polynomial of degree at most m which best approximates f in $L_2(T)$.

(2) (Convergence) Given $f\in L_2(T)$ we have $s_m(f)\to f$ in $L_2(T)$.

(3) (Parseval's Identity) Given $f\in L_2(T)$ we have $\|f\|_2^2=2\pi\sum\limits_{n=-\infty}^{\infty}\left|c_n(f)\right|^2$.

(4) (Inner Product Formula) Given $f,g\in L_2(T)$ we have $\langle f,g\rangle=2\pi\sum\limits_{n=-\infty}^{\infty}c_n(f)\,\overline{c_n(g)}$.

(5) (The Riesz-Fischer Theorem) Given $c_n\in\mathbb{C}$, if $\sum\limits_{n=-\infty}^{\infty}|c_n|^2<\infty$ then there exists a unique $f\in L_2(T)$ such that $c_n=c_n(f)$.

Proof: These are immediate consequences of Theorems 4.23 and 4.24.

5.28 Exercise: Show that when $f\in L_2(T,\mathbb{R})$, Parseval's Identity becomes

$$\|f\|_2^2=2\pi\left|a_0(f)\right|^2+\pi\sum_{n=1}^{\infty}\left|a_n(f)\right|^2+\pi\sum_{n=1}^{\infty}\left|b_n(f)\right|^2.$$

5.29 Exercise: Use Parseval's Identity, together with the result of Example 5.4, to prove that $\sum\limits_{k=0}^{\infty}\frac{1}{(2k+1)^4}=\frac{\pi^4}{96}$ and use this result to calculate $\sum\limits_{n=1}^{\infty}\frac{1}{n^4}$.

5.30 Note: Let $f \in L_1(T)$. Then

$$\begin{aligned} s_m(f)(x) = \sum_{n=-m}^{m} c_n(f)\, e^{inx} &= \sum_{n=-m}^{m} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt \right) e^{inx} \\ = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{n=-m}^{m} e^{in(x-t)}\, dt &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_m(x-t)\, dt \end{aligned}$$

where

$$\begin{aligned} D_m(u) = \sum_{n=-m}^{m} e^{inu} = e^{-imu} \frac{e^{i(2m+1)u} - 1}{e^{iu} - 1} &= \frac{e^{i(m+1)u} - e^{-imu}}{e^{iu} - 1} \cdot \frac{e^{-iu/2}}{e^{-iu/2}} \\ = \frac{e^{i(2m+1)u/2} - e^{i(2m+1)u/2}}{e^{iu/2} - e^{-iu/2}} &= \frac{\sin \frac{(2m+1)u}{2}}{\sin \frac{u}{2}} \,. \end{aligned}$$

5.31 Definition: The above function $D_m : T \to \mathbb{R}$ is called the m^{th} **Dirichlet kernel**.

5.32 Remark: For $f, g \in L_1(T)$, the **convolution** of f with g is the function $f \star g : T \to \mathbb{R}$ given by $(f \star g)(x) = \frac{1}{2\pi} \int_T f(t) g(x-t)\, dt$. Using this notation we have $s_m(f) = f \star D_m$.

5.33 Note: We have

$$\int_{-\pi}^{\pi} D_m(u)\, du = \int_{-\pi}^{\pi} \sum_{n=-m}^{m} e^{inu}\, du = \int_{-\pi}^{\pi} 1 + \sum_{n=1}^{m} 2\cos(nu)\; du = 2\pi$$

and

$$\begin{aligned} \int_{-\pi}^{\pi} \big|D_m(u)\big|\, du &= \int_{-\pi}^{\pi} \left| \frac{\sin \frac{(2m+1)u}{2}}{\sin \frac{u}{2}} \right| du = 2 \int_{0}^{\pi} \left| \frac{\sin \frac{(2m+1)u}{2}}{\sin \frac{u}{2}} \right| du \\ &\geq 2 \int_{u=0}^{\pi} \frac{\left| \sin \frac{(2m+1)u}{2} \right|}{\frac{u}{2}}\, du = 2 \int_{t=0}^{(m+\frac{1}{2})\pi} \frac{|\sin t|}{\frac{t}{2m+1}} \cdot \tfrac{2}{2m+1}\, dt \\ &\geq 4 \sum_{n=1}^{m} \int_{(n-1)\pi}^{n\pi} \frac{|\sin t|}{t}\, dt \geq 4 \sum_{n=1}^{m} \int_{(n-1)\pi}^{n\pi} \frac{|\sin t|}{n\pi}\, dt \\ &= \tfrac{8}{\pi} \sum_{n=1}^{m} \tfrac{1}{n} \geq \tfrac{8}{\pi} \int_{x=1}^{m+1} \tfrac{1}{x}\, dx = \tfrac{8}{\pi} \ln(m+1) \geq \tfrac{8}{\pi}\, \ln m \,. \end{aligned}$$

5.34 Theorem: *(Pointwise Divergence) Let $C(T) = C(T, \mathbb{C})$ be the Banach space of continuous functions $f : T \to \mathbb{C}$ equipped with the supremum norm. There exists a dense $\mathcal{G}_\delta$ set $E \subseteq C(T)$ such that for every $f \in E$ the set of points $x \in T$ at which the Fourier series for f diverges is dense in T.*

Proof: First we fix $x = 0$. For $m \in \mathbb{Z}^+$, define $F_m : C(T) \to \mathbb{C}$ by

$$F_m(f) = s_m(f)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, D_m(t)\, dt.$$

Note that

$$\big|F_m(f)\big| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|f(t)\big|\,\big|D_m(t)\big|\, dt \le \frac{1}{2\pi}\,\|f\|_\infty \int_{-\pi}^{\pi} \big|D_m(t)\big|\, dt$$

so we have

$$\|F_m\|_{op} \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|D_m(t)\big|\, dt\,.$$

We claim that in fact $\|F_m\|_{op} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \big|D_m(t)\big|\, dt$. Fix m and define

$$s(t) = \begin{cases} 1 \text{ if } D_m(t) \ge 0, \\ -1 \text{ if } D_m(t) < 0. \end{cases}$$

Construct continuous functions $g_n : T \to \mathbb{R}$ with $|g_n| \le 1$ such that $g_n \to s$ pointwise. By the Dominated Convergence Theorem, we have

$$F_m(g_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(t) D_m(t)\, dt \longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} s(t) D_m(t)\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|D_m(t)\big|\, dt$$

so that $\|F_m\|_{op} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \big|D_m(t)\big|\, dt$, as claimed. By Note 5.33 we have $\|F_m\|_{op} \ge \frac{4 \ln m}{\pi^2}$, so the set of linear operators $S = \big\{F_m \big| m \in \mathbb{Z}^+\big\}$ is not uniformly bounded. By the Uniform Boundedness Principle, applied to the set S, there exists a function $f \in C(T)$ such that for all $M > 0$ we have $\big|F_m(f)\big| > M$, that is $\big|s_m(f)(0)\big| > M$, for some $m \in \mathbb{Z}^+$. For this function $f \in C(T)$, the Fourier series for f diverges at 0 because $\limsup\limits_{m\to\infty} \big|s_m(f)(0)\big| = \infty$.

Let $Q = \{a_1, a_2, a_3, \cdots\}$ be a dense subset of $[0, 2\pi]$ and consider each a_n as an element in T. For each $n \in \mathbb{Z}^+$ let $f_n(x) = f(x - a_n)$ so that $\limsup\limits_{m\to\infty} \big|s_m(f_n)(a_n)\big| = \infty$. For $n, m \in \mathbb{Z}^+$, define $L_{n,m} : C(T) \to \mathbb{C}$ by $L_{n,m}(f) = s_m(f)(a_n)$. By Condensation of Singularities, the set

$$E = \Big\{ f \in C(T) \,\Big|\, \limsup_{m\to\infty} \big\|L_{n,m}(f)\big\| = \infty \text{ for all } n \in \mathbb{Z}^+ \Big\}$$

is a dense $\mathcal{G}_\delta$ in the Banach space $C(T)$. For each $f \in E$, we have $\limsup\limits_{m \ge 0} \big|s_m(f)(a_n)\big| = \infty$ for every $n \in \mathbb{Z}^+$, so the Fourier series for f diverges at every point a_n.

5.35 Theorem: *(Cesàro Convergence) Let $a_n \in \mathbb{C}$ for $n \geq 0$, let $s_m = \sum_{n=0}^{m} a_n$ and let*

$$\sigma_\ell = \tfrac{1}{\ell+1} \sum_{m=0}^{\ell} s_m \,.$$

If the sequence $\{s_m\}$ converges then so does the sequence $\{\sigma_\ell\}$ and, in this case, we have

$$\lim_{\ell\to\infty} \sigma_\ell = \lim_{m\to\infty} s_m.$$

Proof: The proof is left as an exercise.

5.36 Definition: For $f \in L_1(T) = L_1(T,\mathbb{C})$, we define the ℓ^{th} **Cesàro mean** of the Fourier series of f to be the function $\sigma_\ell(f) : T \to \mathbb{C}$ given by

$$\sigma_\ell(f) = \tfrac{1}{\ell+1} \sum_{m=0}^{\ell} s_m(f).$$

5.37 Note: For $f \in L_1(T) = L_1(T,\mathbb{C})$ we have

$$\begin{aligned}\sigma_\ell(f)(x) &= \frac{1}{\ell+1}\sum_{m=0}^{\ell} s_m(f)(x) = \frac{1}{\ell+1}\sum_{m=0}^{\ell}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_m(x-t)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, \tfrac{1}{\ell+1}\sum_{m=0}^{\ell} D_m(x-t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_\ell(x-t)\,dt\end{aligned}$$

where

$$\begin{aligned}K_\ell(u) &= \frac{1}{\ell+1}\sum_{m=0}^{\ell} D_m(u) = \frac{1}{\ell+1}\sum_{m=0}^{\ell}\frac{\sin\frac{(2m+1)u}{2}}{\sin\frac{u}{2}} \\ &= \frac{1}{(\ell+1)\sin\frac{u}{2}}\cdot \mathrm{Im}\Big(\sum_{m=0}^{\ell} e^{i(2m+1)u/2}\Big) = \frac{1}{(\ell+1)\sin\frac{u}{2}}\cdot \mathrm{Im}\Big(e^{iu/2}\sum_{m=0}^{\ell} e^{imu}\Big) \\ &= \frac{1}{(\ell+1)\sin\frac{u}{2}}\cdot \mathrm{Im}\Big(e^{iu/2}\,\frac{e^{i(\ell+1)u}-1}{e^{iu}-1}\Big) = \frac{1}{(\ell+1)\sin\frac{u}{2}}\cdot \mathrm{Im}\Big(\frac{e^{i(\ell+1)u}-1}{e^{iu/2}-e^{\frac{-iu}{2}}}\Big) \\ &= \frac{1}{(\ell+1)\sin\frac{u}{2}}\cdot \mathrm{Im}\Big(\frac{e^{i(\ell+1)u/2}-e^{-i(\ell+1)u/2}}{e^{iu/2}-e^{-iu/2}}\cdot e^{i(\ell+1)u/2}\Big) \\ &= \frac{1}{(\ell+1)\sin\frac{u}{2}}\cdot\frac{\sin\frac{(\ell+1)u}{2}}{\sin\frac{u}{2}}\cdot \sin\frac{(\ell+1)u}{2} = \frac{\sin^2\frac{(\ell+1)u}{2}}{(\ell+1)\,\sin^2\frac{u}{2}}\,.\end{aligned}$$

5.38 Definition: The above function $K_\ell : T \to \mathbb{R}$ is called the ℓ^{th} **Féjer kernel**.

5.39 Remark: Using convolution notation, for $f \in L_1(T)$ we have $\sigma_\ell(f) = f{\star}K_\ell$.

5.40 Lemma: *We have*

(1) For $0 < t \leq \pi$ we have $0 \leq K_\ell(t) \leq \frac{\pi^2}{(\ell+1)\,t^2}$.

(2) $\displaystyle\int_{-\pi}^{\pi} K_\ell(t)\,dt = 2\int_0^{\pi} K_\ell(t)\,dt = 2\pi$.

(3) $\displaystyle\int_{\pi}^{\pi} f(t)K_\ell(x-t)\,dt = \int_{-\pi}^{\pi} f(x+t)K_\ell(t)\,dt = \int_{-\pi}^{\pi} f(x-t)K_\ell(t)\,dt$.

Proof: The proof is left as an exercise.

5.41 Theorem: *(Convergence of the Cesàro Means) Let $f \in L_1(T)$ and consider f as a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$.*

(1) If $a \in \mathbb{R}$ and the one-sided limits $f(a^-) = \lim_{x \to a^-} f(x)$ and $f(a^+) = \lim_{x \to a^+} f(x)$ both exist in $\mathbb{C}$, then

$$\lim_{\ell \to \infty} \sigma_\ell(f)(a) = \frac{f(a^-) + f(a^+)}{2}\,.$$

(2) If $a, b \in \mathbb{R}$ with $a \leq b$ and f is continuous in $[a, b]$ then $\sigma_\ell \to f$ uniformly on $[a, b]$.

Proof: By Part 3 of the above lemma, we have

$$\sigma_\ell(f)(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) K_\ell(a - t)\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(a + t) + f(a - t)}{2} K_\ell(t)\, dt$$

and by Part 2 of the above lemma we have

$$\frac{f(a^+) + f(a^-)}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(a^+) + f(a^-)}{2} K_\ell(t)\, dt$$

and so

$$\begin{aligned}
\left|\sigma_\ell(f)(a) - \frac{f(a^+) + f(a^-)}{2}\right| &= \left|\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{f(a{+}t) + f(a{-}t)}{2} - \frac{f(a^+) + f(a^-)}{2}\right) K_\ell(t)\ dt\right| \\
&= \left|\frac{1}{2\pi} \int_0^{\pi} \Big((f(a{+}t) - f(a^+)) + (f(a{-}t) - f(a^-))\Big) K_\ell(t)\ dt\right| \\
&\leq \frac{1}{2\pi} \int_0^{\pi} \Big(\big|f(a{+}t) - f(a^+)\big| + \big|f(a{-}t) - f(a^-)\big|\Big) K_\ell(t)\ dt \\
&= I_\delta + J_\delta,
\end{aligned}$$

for any $0 < \delta \leq \pi$, where

$$\begin{aligned}
I_\delta &= \frac{1}{2\pi} \int_0^{\delta} \Big(\big|f(a{+}t) - f(a^+)\big| + \big|f(a{-}t) - f(a^-)\big|\Big) K_\ell(t)\ dt \\
J_\delta &= \frac{1}{2\pi} \int_\delta^{\pi} \Big(\big|f(a{+}t) - f(a^+)\big| + \big|f(a{-}t) - f(a^-)\big|\Big) K_\ell(t)\ dt\,.
\end{aligned}$$

Let $\epsilon > 0$. Choose $\delta > 0$ so that for all $0 < t < \delta$ we have $\big|f(a + t) - f(a^+)\big| < \frac{\epsilon}{2}$ and $\big|f(a - t) - f(a^-)\big| < \frac{\epsilon}{2}$. Then, by Part 2 of the above lemma,

$$I_\delta \leq \frac{1}{2\pi} \int_0^{\pi} \epsilon \cdot K_\ell(t)\, dt \leq \tfrac{\epsilon}{2}.$$

By Part 1 of the above lemma, for $\delta \leq t \leq \pi$ we have $K_\ell(t) \leq \frac{\pi^2}{(\ell+1)\,\delta^2}$ so for $\ell + 1 \geq \frac{M}{\epsilon}$ where $M = \pi\big(\|f\|_1 + \pi|f(a^+)| + \pi|f(a^-)|\big)/\delta^2$ we have

$$\begin{aligned}
J_\delta &\leq \frac{1}{2\pi} \int_\delta^{\pi} \Big(|f(a + t)| + |f(a - t)| + |f(a^+)| + |f(a^-)|\Big) \tfrac{\pi^2}{(\ell+1)\,\delta^2}\ dt \\
&\leq \tfrac{1}{2\pi} \cdot \tfrac{\pi^2}{(\ell+1)\,\delta^2} \big(\|f\|_1 + \pi|f(a^+)| + \pi|f(a^-)|\big) = \tfrac{M}{2(\ell+1)} \leq \tfrac{\epsilon}{2}.
\end{aligned}$$

This proves Part (1), and Part (2) can be proven using the same method noting that the estimates can be made uniformly.

5.42 Corollary: *Let $f \in L_1(T)$, consider f as a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$, and let $a \in \mathbb{R}$. If $f(a^+)$, $f(a^-)$ and $\lim_{m\to\infty} s_m(f)(a)$ all exist in $\mathbb{C}$ then*

$$\lim_{m\to\infty} s_m(f)(a) = \frac{f(a^+) + f(a^-)}{2}.$$

5.43 Remark: The above corollary justifies the argument given in Remark 5.5 where we showed that $\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$.